A Quantile Approach to Evaluating Asset Pricing Models

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Outline

How to Evaluate Asset Pricing Models?

- A central question in finance: what drives $\mathbb{E}_t\left[R_{m,t\to N}\right] - R_{f,t\to N}$?
  - $R_{m,t\to N}$ is the return on the market (S&P500) from t to $t+N$
  - $R_{f,t\to N}$ is risk-free rate, observed at time t
  - $\mathbb{E}_t\left[\cdot\right]$ is the *conditional* expectation
- Misspecification benchmark: How well does an asset pricing model match the *mean* or *variance* of the market return
- This paper: use quantiles to analyze misspecification in the *entire* distribution
  - Which segment of the return distribution in the asset pricing model is most misspecified
  - Model-free and accounts for conditional information
  - Highlights the importance of *disaster risk*

Literature Review

- *Misspecification of asset pricing models*: Hansen and Jagannathan (1991), Stutzer (1995), Bansal and Lehmann (1997), Alvarez and Jermann (2005), Almeida and Garcia (2012), Liu (2021)
- *(Time varying) disaster risk*: Rietz (1988), Barro (2006), Weitzman (2007), Bollerslev and Todorov (2011), Gabaix (2012), Wachter (2013), Isoré and Szczerbowicz (2017), Farhi and Gourio (2018), Seo and Wachter (2019)
- *Recovering forward looking beliefs*: Ross (2015), Borovička, Hansen, and Scheinkman (2016), Martin (2017), Qin and Linetsky (2017), Martin and Wagner (2019), Schneider and Trojani (2019), Chabi-Yo and Loudis (2020)
- *Nonparametrcic SDF estimation*: Aït-Sahalia and Lo (1998), Jackwerth (2000), Rosenberg and Engle (2002), Beare and Schmidt (2016), Linn, Shive, and Shumway (2018), Cuesdeanu and Jackwerth (2018)

Framework

- Returns evolve according to $R_{m,t\to N} \sim \mathbb{P}_t$ (*physical measure*)
- No-arbitrage assumption: there exists $\widetilde{\mathbb{P}}_t$ (*risk-neutral measure*), such that

$$\widetilde{\mathbb{E}}_t\left(R_{m,t\to N}\right) = R_{f,t\to N}$$

- Differences between $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ induce a risk premium:

$$\mathbb{E}_t\left[R_{m,t\to N}\right] - R_{f,t\to N} = \mathbb{E}_t\left[R_{m,t\to N}\right] - \widetilde{\mathbb{E}}_t\left(R_{m,t\to N}\right) > 0 \qquad (1)$$

- Suppose we know $\mathbb{P}_t$, then there are many $\widetilde{\mathbb{P}}_t$ consistent with (1)
- In this sense, evaluating a model based on explaining (1) may not give a complete picture

Quantile Approach

- Let $Q_{t,\tau}$ and $\widetilde{Q}_{t,\tau}$ denote the quantile functions of the physical and risk-neutral measures

$$\tau = \mathbb{P}_t(R_{m,t\to N} \leq Q_{t,\tau}) =: F_t(Q_{t,\tau})$$
$$\tau = \widetilde{\mathbb{P}}_t(R_{m,t\to N} \leq \widetilde{Q}_{t,\tau}) =: \widetilde{F}_t(\widetilde{Q}_{t,\tau}) \quad \text{for all } \tau \in (0,1)$$

- This paper considers $Q_{t,\tau} - \widetilde{Q}_{t,\tau}$
  - A local discrepancy measure between $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ Figure
- Fundamental difficulty: $Q_{t,\tau}$ is not observed from asset prices
- This paper: infers $Q_{t,\tau} - \widetilde{Q}_{t,\tau}$ indirectly from *observed* asset prices
  - Robust to model misspecification
  - Accounts for conditional information
- Model-free approach is important. For example, disaster risk models are notoriously hard to estimate

Econometric Model

- To measure $Q_{t,\tau} - \widetilde{Q}_{t,\tau}$, I consider

$$\underbrace{Q_{t,\tau}(R_{m,t\to N})}_{\text{Unobserved}} = \beta_0(\tau) + \beta_1(\tau)\,\underbrace{\widetilde{Q}_{t,\tau}(R_{m,t\to N})}_{\text{Observed}}, \qquad \forall \tau \in (0,1)$$

- Benchmark: if the world is risk-neutral, $\beta_0(\tau) = 0$ and $\beta_1(\tau) = 1$ for all τ
- More generally, departures from $[\beta_0(\tau), \beta_1(\tau)] = [0, 1]$ reflect a *local discrepancy* between $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ at τ^{th} percentile
- Estimate the model by quantile regression

$$[\widehat{\beta}_0(\tau), \widehat{\beta}_1(\tau)] = \underset{(\beta_0,\beta_1)\in\mathbb{R}^2}{\arg\min} \sum_{t=1}^{T} \rho_\tau(R_{m,t\to N} - \beta_0 - \beta_1 \widetilde{Q}_{t,\tau}),$$

where $\rho_\tau(x) = x(\tau - \mathbb{1}(x < 0))$ Figure

Estimation

- Recall $[\widehat{\beta}_0(\tau), \widehat{\beta}_1(\tau)] = \arg\min_{(\beta_0,\beta_1)\in\mathbb{R}^2} \sum_{t=1}^{T} \rho_\tau(R_{m,t\to N} - \beta_0 - \beta_1 \widetilde{Q}_{t,\tau})$
- $\widetilde{Q}_{t,\tau}$ is observed in from asset prices since

$$\widetilde{F}_t\left(\frac{K}{S_t}\right) = R_{f,t\to N}\frac{\partial}{\partial K}\mathrm{Put}_t(K)$$

- $\mathrm{Put}_t(K)$ is the price of a (European) put option with payoff $\max(K - R_{m,t\to N}, 0)$ Figure
- The insight is $R_{m,t\to N} \sim \mathbb{P}_t$
  - Quantile regression estimates best linear approximation to $Q_{t,\tau}$
  - Similar to how OLS estimates best linear approximation to $\mathbb{E}_t\left[R_{m,t\to N}\right]$
- Key benefit: only use observed data $\{R_{m,t\to N}, \widetilde{Q}_{t,\tau}\}_{t=1}^{T}$

- To evaluate how close $Q_{t,\tau}$ is to $\widetilde{Q}_{t,\tau}$, I use two other measures of fit
- $R^1(\tau)$, analogue of OLS R^2:

$$R^1(\tau) := 1 - \frac{\min_{b_0,b_1} \sum_{t=1}^{T} \rho_\tau(R_{m,t\to N} - b_0 - b_1 \widetilde{Q}_{t,\tau})}{\min_{b_0} \sum_{t=1}^{T} \rho_\tau(R_{m,t\to N} - b_0)}$$

- $R^1_{oos}(\tau)$, analogue of OLS out-of-sample R^2:

$$R^1_{oos}(\tau) := 1 - \frac{\sum_{t=w}^{T} \rho_\tau(R_{m,t\to N} - \widetilde{Q}_{t,\tau})}{\sum_{t=w}^{T} \rho_\tau(R_{m,t\to N} - \overline{Q}_{t,\tau})},$$

where w is the initial sample size and $\overline{Q}_{t,\tau}$ the historical rolling quantile

Data and Empirical Results

- For $R_{m,t\to N}$, I use overlapping 30-day returns on the S&P500 from 2003–2021
- For $\widetilde{Q}_{t,\tau}$, I use (European) Put and Call option data on the S&P500 over the same period
- Table shows that $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ are similar in the right-tail, but not in the left-tail
- Model-free evidence for disaster risk
- Results are similar at longer horizons (60 and 90 days)

Empirical Results

Horizon	τ	$\widehat{\beta}_0(\tau)$	$\widehat{\beta}_1(\tau)$	Wald test (p-value)	$R^1(\tau)[\%]$	$R^1_{oos}(\tau)[\%]$	$\overline{\text{Hit}}[\%]$
30 days*	0.05	0.43 (0.220)	0.56 (0.235)	0.01	6.28	6.11	-2.67 (0.699)
	0.1	0.45 (0.244)	0.54 (0.254)	0.03	3.45	1.01	-3.56 (1.162)
	0.2	0.69 (0.375)	0.30 (0.382)	0.10	0.55	0.89	-3.73 (1.695)
	0.5	-0.60 (0.307)	1.61 (0.305)	0.00	1.65	2.24	-8.07 (2.567)
	0.8	-0.09 (0.163)	1.09 (0.158)	0.23	12.44	12.50	-3.24 (2.221)
	0.9	0.03 (0.113)	0.97 (0.108)	0.96	20.41	21.88	-0.04 (1.509)
*(Obs. 4333)	0.95	0.12 (0.119)	0.89 (0.113)	0.51	27.07	31.31	0.27 (1.120)

Interpreting the Results

- To match the data, an asset pricing model should imply distinct physical and risk-neutral measures in the left tail Figure
- Equity premium is driven by disaster risk

$$\begin{aligned}\mathbb{E}_t\left[R_{m,t\to N}\right] - R_{f,t\to N} &= \int_0^1 \left(Q_{t,\tau} - \widetilde{Q}_{t,\tau}\right) \mathrm{d}\tau \\ &= \underbrace{\int_0^{\underline{\tau}} \left(Q_{t,\tau} - \widetilde{Q}_{t,\tau}\right) \mathrm{d}\tau}_{\text{disaster risk}} + \int_{\underline{\tau}}^1 \left(Q_{t,\tau} - \widetilde{Q}_{t,\tau}\right) \mathrm{d}\tau\end{aligned}$$

- Recovery theorem: $\mathbb{P}_t \approx \widetilde{\mathbb{P}}_t$ in the right-tail, and $\widetilde{\mathbb{P}}_t$ was obtained model-free

Comparison with Existing Estimation Methods

- An alternative approach nonparametrically estimates $M_t(x) := \widetilde{f}_t(x)/f_t(x)$, where $\widetilde{f}_t$ and f_t denote the risk-neutral and physical density
- Aït-Sahalia and Lo (1998), Jackwerth (2000) and Rosenberg and Engle (2002) estimate an average pricing kernel
  - hard to account for conditional information

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- Aït-Sahalia and Lo (1998), Jackwerth (2000) and Rosenberg and Engle (2002) estimate an average pricing kernel
  - hard to account for conditional information
- Linn, Shive, and Shumway (2018) and Cuesdeanu and Jackwerth (2018) use sieve estimation that accounts for conditional information
  - unclear which basis functions to choose and how many
  - optimization is non-convex; sometimes undefined
  - estimated $f_t(x)$ cannot change shape
  - in simulation, quantile regression is much more precise
- In addition, for quantile regression
  - no need to estimate $f_t(x)$
  - requires only $\widetilde{Q}_{t,\tau}$, not $\widetilde{f}_t(x)$

Conditional Lognormal Model

- Consider a Black-Scholes type model

$$R_{m,t\to N} = \exp([\mu - \frac{1}{2}\sigma_t^2] + \sigma_t Z_{t+1}), \qquad \mathbb{E}_t\left[R_{m,t\to N}\right] = e^{\mu} \qquad \text{under } \mathbb{P}_t$$

$$R_{m,t\to N} = \exp([r_f - \frac{1}{2}\sigma_t^2] + \sigma_t Z_{t+1}), \qquad \widetilde{\mathbb{E}}_t\left(R_{m,t\to N}\right) = e^{r_f} \qquad \text{under } \widetilde{\mathbb{P}}_t$$

$$\Longrightarrow \; M_t(x) \coloneqq \frac{\widetilde{f}_t(x)}{f_t(x)} = C_t x^{-\left(\frac{\mu - r_f}{\sigma_t}\right)},$$

where σ_t is conditional volatility and $Z_{t+N} \sim \mathcal{N}(0,1)$

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$$R_{m,t\to N} = \exp([r_f - \frac{1}{2}\sigma_t^2] + \sigma_t Z_{t+1}), \qquad \widetilde{\mathbb{E}}_t\left(R_{m,t\to N}\right) = e^{r_f} \qquad \text{under } \widetilde{\mathbb{P}}_t$$

$$\implies M_t(x) := \frac{\widetilde{f}_t(x)}{f_t(x)} = C_t x^{-\left(\frac{\mu - r_f}{\sigma_t}\right)},$$

where σ_t is conditional volatility and $Z_{t+N} \sim \mathcal{N}(0,1)$

- One can show: $Q_{t,\tau} = e^{(\mu - r_f)}\widetilde{Q}_{t,\tau}$, and quantile regression will find it
- Sieve estimation yields an estimate $M_t(x) = C_t g(x)$, where $g(x)$ is time invariant
  - Does not account for time variation in σ_t; shape is time invariant
- The paper generalizes to time varying μ_t and $r_{f,t}$

Approximating the Left-tail

- Recent papers argue that *time-varying* disaster risk is important to explain expected returns
- Definition of time-varying disaster risk typically depends on parameters in a model
- As a general definition, I adopt $Q_{t,\tau}$ at $\tau = 0.05$ to represent disaster risk

Approximating the Left-tail

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- Definition of time-varying disaster risk typically depends on parameters in a model
- As a general definition, I adopt $Q_{t,\tau}$ at $\tau = 0.05$ to represent disaster risk
- Previous results suggest $Q_{t,\tau} \gg \widetilde{Q}_{t,\tau}$ when $\tau = 0.05$, so cannot use the risk-neutral quantile in this region
- Is there a way to approximate $Q_{t,\tau} - \widetilde{Q}_{t,\tau}$ in the left-tail?

Risk-adjustment in the Left-tail

- Some functional analysis gives (von-Mises calculus) Figure

$$Q_{t,\tau} - \widetilde{Q}_{t,\tau} = \underbrace{\frac{\tau - F_t(\widetilde{Q}_{t,\tau})}{\widetilde{f}_t(\widetilde{Q}_{t,\tau})}}_{\text{unobserved risk-adjustment}} + \underbrace{o\left(\left\|F_t - \widetilde{F}_t\right\|\right)}_{\text{small under no near-arbitrage}}$$

- Under certain conditions, I show

Theorem

$$\tau - F_t\left(\widetilde{Q}_{t,\tau}\right) \geq \frac{\sum_{k=1}^{3} \frac{(-1)^{k+1}}{R_{f,t\to N}^{k}} \left(\tau \widetilde{\mathbb{M}}_{t\to N}^{(k)} - \widetilde{\mathbb{M}}_{t\to N}^{(k)}[\widetilde{Q}_{t,\tau}]\right)}{1 + \sum_{k=1}^{3} \frac{(-1)^{k+1}}{R_{f,t\to N}^{k}} \widetilde{\mathbb{M}}_{t\to N}^{(k)}} =: \mathrm{LB}_{t,\tau},$$

where

$$\widetilde{\mathbb{M}}_{t\to N}^{(n)} := \widetilde{\mathbb{E}}_t\left[(R_{m,t\to N} - R_{f,t\to N})^n\right]$$
$$\widetilde{\mathbb{M}}_{t\to N}^{(n)}[k_0] := \widetilde{\mathbb{E}}_t\left[\mathbb{1}\left(R_{m,t\to N} \leq k_0\right)(R_{m,t\to N} - R_{f,t\to N})^n\right]$$

Risk-adjustment in the Left-tail

- Combining the previous results renders an *observed* inequality

$$Q_{t,\tau} - \widetilde{Q}_{t,\tau} \geq \overbrace{\frac{\mathrm{LB}_{t,\tau}}{\widetilde{f}_t(\widetilde{Q}_{t,\tau})}}^{\textit{risk-adjustment}} =: \mathrm{RA}_{t,\tau}$$

- If the inequality is tight, we have an observed measure of time-varying disaster risk
- To test this, I form quantile adjusted returns, $R_{m,t\to N} - \widetilde{Q}_{t,\tau}$
- Then use quantile regression to estimate

$$Q_{t,\tau}(R_{m,t\to N}) - \widetilde{Q}_{t,\tau}(R_{m,t\to N}) = \beta_0(\tau) + \beta_1(\tau)\mathrm{RA}_{t,\tau}$$

- A tight lower bound suggests $\beta_0(\tau) = 0$ and $\beta_1(\tau) = 1$ for all τ

Quantile Regression Lower Bound

	Horizon (in days)	$\widehat{\beta}_0(\tau)$	$\widehat{\beta}_1(\tau)$	Wald test (p-value)	$R^1(\tau)[\%]$	Obs
$\tau = 0.05$	30	-0.01 (0.006)	4.43 (0.350)	0.00	6.03	4333
	60	-0.01 (0.018)	5.53 (0.717)	0.00	3.60	4312
	90	-0.02 (0.040)	6.37 (1.385)	0.00	4.91	4291
$\tau = 0.1$	30	-0.01 (0.006)	2.17 (0.420)	0.02	3.18	4333
	60	-0.02 (0.014)	3.25 (0.602)	0.00	2.23	4312
	90	-0.02 (0.024)	3.05 (0.703)	0.00	4.43	4291
$\tau = 0.2$	30	-0.01 (0.006)	1.33 (0.418)	0.03	0.41	4333
	60	-0.02 (0.013)	1.50 (0.506)	0.49	0.48	4312
	90	-0.02 (0.022)	1.36 (0.694)	0.76	1.46	4291

- The lower bound is not tight, but a highly relevant predictor
- Can be interpeted as a quantile factor model:

$$Q_{t,\tau} = \widetilde{Q}_{t,\tau} + \beta(\tau)\mathrm{RA}_{t,\tau}$$

- I also consider a direct quantile forecast without estimation

$$\widehat{Q}_{t,\tau} = \widetilde{Q}_{t,\tau} + \mathrm{RA}_{t,\tau}$$

$$Q_{t,\tau}(R_{m,t\to N}) = \beta_0(\tau) + \beta_1(\tau)\widehat{Q}_{t,\tau}$$

- Point estimates are closer to the [0, 1]-benchmark and $R^1_{oos}(\tau)$ is higher relative to using $\widetilde{Q}_{t,\tau}$ only
- In sum, $\mathrm{RA}_{t,\tau}$ is a good proxy for $Q_{t,\tau}$ in the left-tail, and hence for disaster risk

Risk-adjusted Quantile Regression

	Horizon (in days)	$\widehat{\beta}_0(\tau)$	$\widehat{\beta}_1(\tau)$	Wald test (p-value)	$R^1(\tau)[\%]$	$R^1_{oos}(\tau)[\%]$	Obs
$\tau = 0.05$	30	0.29 (0.283)	0.70 (0.301)	0.13	6.28	9.94	4333
	60	0.30 (0.434)	0.71 (0.484)	0.06	3.40	17.81	4312
	90	0.36 (0.594)	0.64 (0.687)	0.10	4.26	21.98	4291
$\tau = 0.1$	30	0.28 (0.310)	0.72 (0.322)	0.27	3.57	4.02	4333
	60	0.38 (0.444)	0.61 (0.472)	0.22	2.35	9.22	4312
	90	0.31 (0.613)	0.70 (0.664)	0.13	4.19	13.22	4291
$\tau = 0.2$	30	0.57 (0.499)	0.43 (0.507)	0.47	0.58	2.53	4333
	60	0.44 (0.617)	0.56 (0.630)	0.40	0.57	4.28	4312
	90	0.23 (0.760)	0.78 (0.774)	0.56	0.70	5.99	4291

Disaster Risk and Peso Problem

- Ross (2015) on disaster risk: "*It is unseen and not directly observable but it exerts a force that can change over time and that can profoundly influence markets*"
- Disaster risk has two components
  - Insurance: If investors pay more to insure against disaster risk, it drives down $\widetilde{Q}_{t,\tau}$
  - Probability: If investors perceive a higher likelihood of a loss, it drives down $Q_{t,\tau}$

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  - Insurance: If investors pay more to insure against disaster risk, it drives down $\widetilde{Q}_{t,\tau}$
  - Probability: If investors perceive a higher likelihood of a loss, it drives down $Q_{t,\tau}$
- Normally these effects cannot be untangled since $Q_{t,\tau}$ is unobserved, but $Q_{t,\tau} \approx \widetilde{Q}_{t,\tau} + \mathrm{RA}_{t,\tau}$ Figure
- Find that both effects are important, but insurance effect is more dominant
  - At the height of the 2008 financial crisis and 2020 Covid crisis, a market return of -28% or lower has a 5% probability
  - 57 times higher than historical estimate

Stochastic Discount Factor

- Foregoing results suggest $Q_{t,\tau} \gg \widetilde{Q}_{t,\tau}$, or $\tau - F_t(\widetilde{Q}_{t,\tau}) \gg 0$
- What does this imply about the stochastic discount factor (SDF), $M_{t\to N}(x) = \widetilde{f}_t(x)/f_t(x)$? Notice $\mathbb{E}_t\left[M_{t\to N} R_{m,t\to N}\right] = 1$
- The SDF is typically the focal point in asset pricing models, as opposed to $\widetilde{F}_t$

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- The SDF is typically the focal point in asset pricing models, as opposed to $\widetilde{F}_t$
- In representative agent models, the SDF can be interpreted as the marginal utility of wealth (or substitution)
- Suppose an agent has CRRA utility with risk-aversion γ and time-discount factor β, then

$$M_{t\to N} = \beta \left(\frac{C_{t+N}}{C_t} \right)^{-\gamma},$$

where C_t and C_{t+N} denote consumption

Hansen-Jagannathan Bound

- In a seminal paper, Hansen and Jagannathan (1991) showed

$$\frac{\sigma_t(M_{t\to N})}{\mathbb{E}_t\left[M_{t\to N}\right]} \geq \frac{\mathbb{E}_t\left[R_{m,t\to N}\right] - R_{f,t\to N}}{\sigma_t(R_{m,t\to N})}$$

- Using historical estimates, the right hand side is about 0.5
- Under IID assumption and power utility, the latter implies $\gamma\sigma(C_{t+N}/C_t) \geq 0.5$
- Since $\sigma(C_{t+N}/C_t)$ is about 1%, we must have $\gamma \geq 50$, a huge level of risk-aversion (*Equity premium puzzle*)

Quantile Version of Hansen-Jagannathan

- Let $\phi_t(\tau) = F_t(Q_{t,\tau})$. If the world is risk-neutral, $\phi_t(\tau) = \tau$.
- Hence, $\tau - \phi_t(\tau)$ reflects *locally* how much $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ differ
- In the spirit of Hansen and Jagannathan (1991), I derive a quantile bound

Proposition (Quantile bound)

Assume no-arbitrage, then

$$\frac{\sigma_t(M_{t\to N})}{\mathbb{E}_t\left[M_{t\to N}\right]} \geq \frac{|\tau - \phi_t(\tau)|}{\sqrt{\phi_t(\tau)(1-\phi_t(\tau))}} \qquad \forall \tau \in (0,1)$$

- Any pointwise difference between $\mathbb{P}_t$ and $\widetilde{\mathbb{P}}_t$ leads to a volatile SDF
- Previous results show that volatility comes from the left-tail

Quantile Bound in Disaster Risk Models

- Consider the disaster risk model in Backus, Chernov, and Martin (2011)

$$\log M_{t+1} = \log \beta - \gamma \log G_{t+1},$$

where $G_{t+1} = C_{t+1}/C_t$ is consumption growth in period $t+1$, and

$$\log G_{t+1} = z_{1,t+1} + z_{2,t+1}, \quad z_{1,t+1} \sim \mathcal{N}(\mu, \sigma^2), \quad z_{2,t+1}|j \sim \mathcal{N}(j\theta, j\delta^2),$$

and j is Poisson

- $z_{2,t+1}$ captures occasional disasters

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and j is Poisson

- $z_{2,t+1}$ captures occasional disasters
- The model pins down $\mathbb{P}$ and M_{t+1}, and as a result $\widetilde{\mathbb{P}}$
- The quantile bound shows the effect on SDF volatility when disasters are introduced ($z_{2,t+1} \equiv 0$ vs. $z_{2,t+1}$ random) Figure

Quantile Bound in the Data

- Estimate the quantile bound on S&P500 returns
- Bound peaks around 5^{th} percentile and is stronger than HJ bound
- Shape is similar to that predicted by disaster risk model Figure
- Additional model-free evidence for disaster risk

Conclusion

- This paper takes a local approach to evaluate asset pricing models
  - Compare quantiles of physical and risk-neutral distribution
  - Quantile regression estimates $Q_{t,\tau} - \widetilde{Q}_{t,\tau}$
- Accounts for time-varying conditional information and is model-free
- Computationally simpler and more accurate than extant methods
- Considers misspecification over the entire distribution, not just mean or variance

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- Accounts for time-varying conditional information and is model-free
- Computationally simpler and more accurate than extant methods
- Considers misspecification over the entire distribution, not just mean or variance
- Find in the left-tail, $Q_{t,\tau}$ and $\widetilde{Q}_{t,\tau}$ are very different, but not in the right-tail
  - Model-free evidence for disaster risk
- Propose a model-free measure of time varying disaster risk
- Repercussions for modeling $\mathbb{P}_t$ and the SDF

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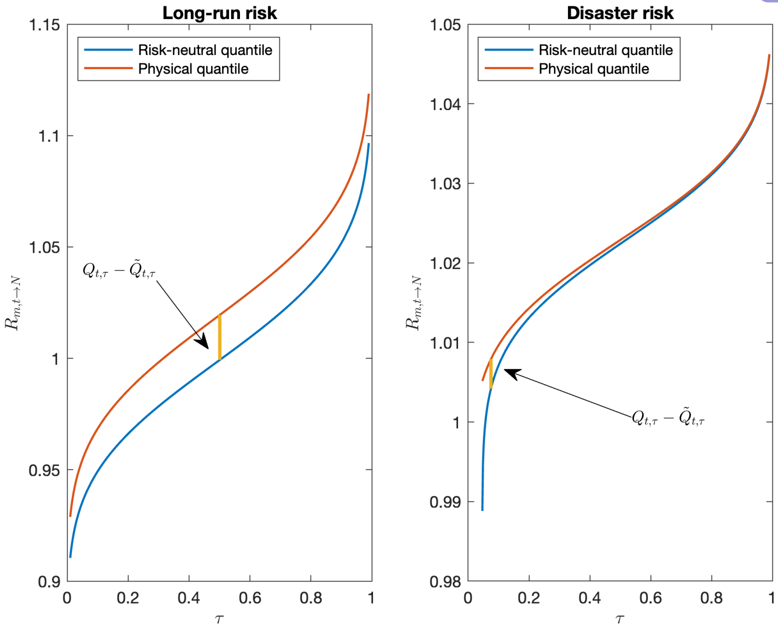
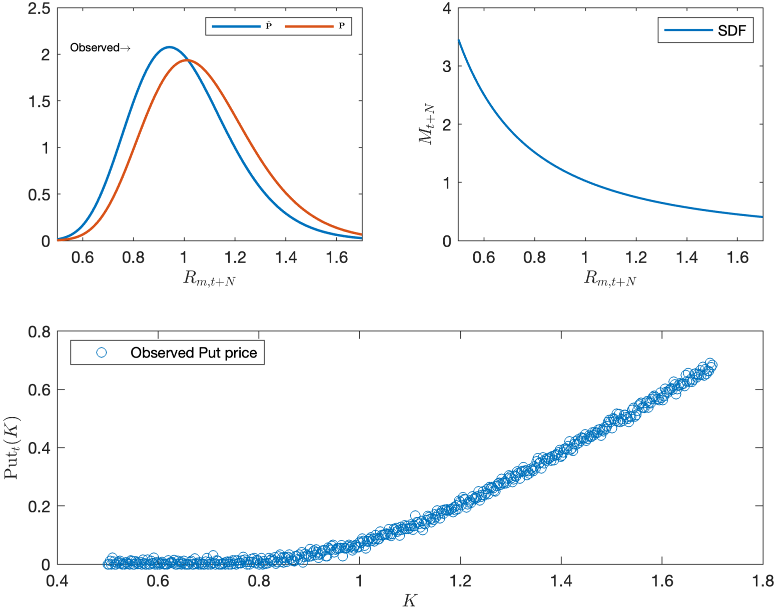


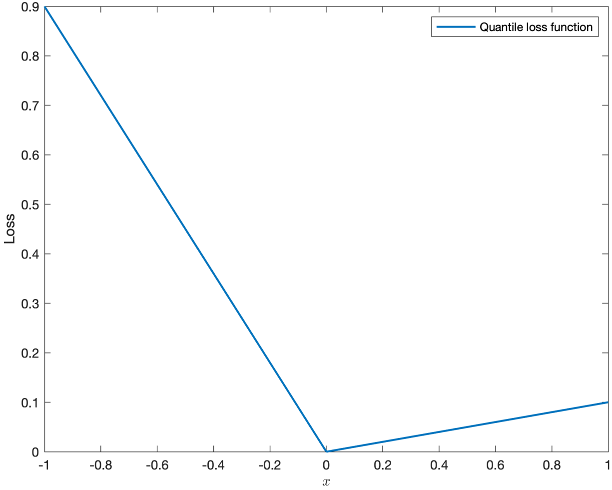
Long-run risk
Risk-neutral quantile
Physical quantile
$Q_{t,\tau} - \tilde{Q}_{t,\tau}$
$R_{m,t\to N}$
τ
1.15
1.1
1.05
1
0.95
0.9
0
0.2
0.4
0.6
0.8
1
Disaster risk
Risk-neutral quantile
Physical quantile
$Q_{t,\tau} - \tilde{Q}_{t,\tau}$
$R_{m,t\to N}$
τ
1.05
1.04
1.03
1.02
1.01
1
0.99
0.98
0
0.2
0.4
0.6
0.8
1

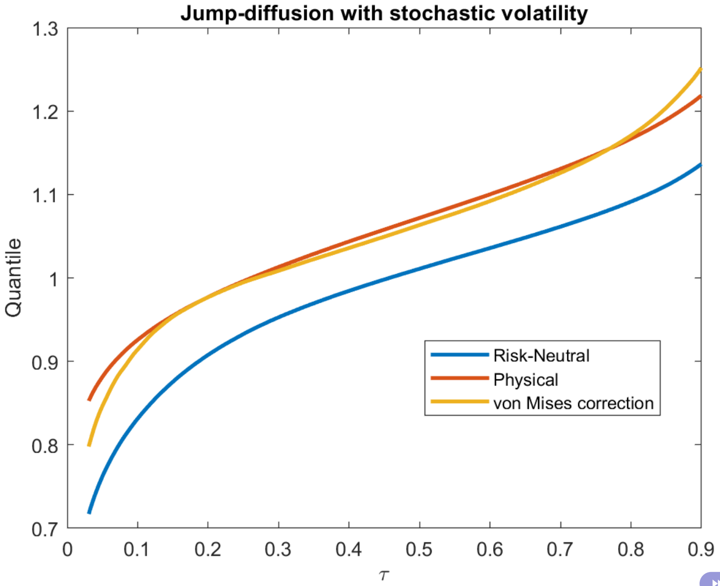
Back


Observed→
P̃
P
$R_{m,t+N}$
SDF
M_{t+N}
$R_{m,t+N}$
Observed Put price
$\mathrm{Put}_t(K)$
K

Quantile Loss Function at $\tau = 0.1$

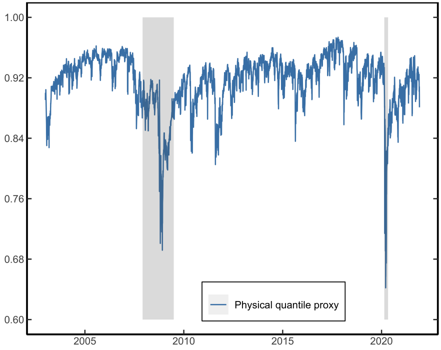
Back



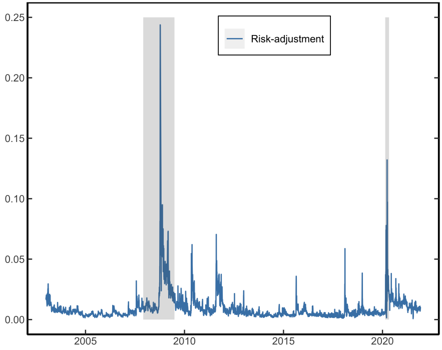

Jump-diffusion with stochastic volatility
Quantile
1.3
1.2
1.1
1
0.9
0.8
0.7
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
τ
Risk-Neutral
Physical
von Mises correction

Back

Time Variation at 5$^{\text{th}}$ Percentile

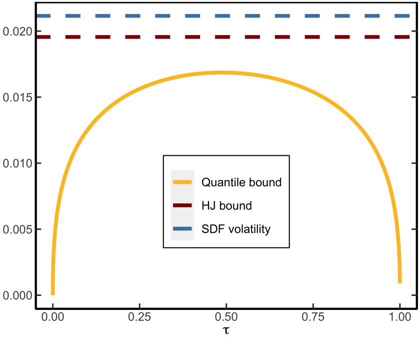


(a) $\widetilde{Q}_{t,\tau} + \text{RA}_{t,\tau}$

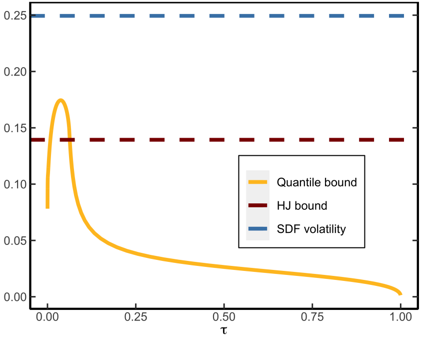


(b) $\text{RA}_{t,\tau} \approx Q_{t,\tau} - \widetilde{Q}_{t,\tau}$

Back

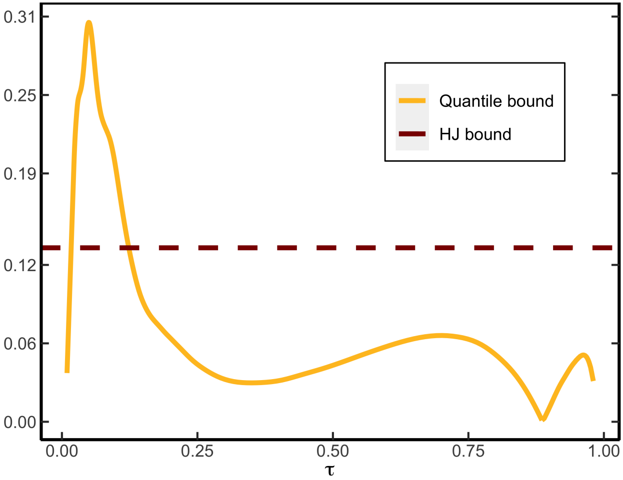


(a) Without disasters



(b) With disasters

Back


0.31
0.25
0.19
0.12
0.06
0.00
0.00
0.25
0.50
0.75
1.00
τ
Quantile bound
HJ bound

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