# Robust Asset-Liability Management 

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#### Abstract

How should financial institutions hedge their balance sheets against interest rate risk when they have long-term assets and liabilities? Using the perspective of functional and numerical analysis, we propose a model-free bond portfolio selection method that generalizes classical immunization and accommodates arbitrary liability structure, portfolio constraints, and perturbations in interest rates. We prove the generic existence of an immunizing portfolio that maximizes the worst-case equity with a tight error estimate and provide a solution algorithm. Numerical evaluations using empirical and simulated yield curves from a no-arbitrage term structure model support the feasibility and accuracy of our approach relative to existing methods.


Keywords: immunization, interest rate risk, maxmin, model-free, robustness.

JEL codes: C65, G11, G12, G22.

## 1 Introduction

Many financial institutions have long-term commitments. For instance, insurance companies promise annuities or life insurance payments to customers; (definedbenefit) pension plans promise predetermined pension payments to retirees; or commercial banks may make long-term loans at fixed interest rates and thus commit to receiving certain future cash flows in exchange of funding the projects with short-term deposits. In such circumstances, it becomes crucial for financial

[^0]institutions to effectively manage their assets and liabilities to hedge against interest rate risk. The recent gilt market crisis in the UK showcases the importance of liability-driven investing strategies and the risk associated with interest rate changes, which eventually led to an $£ 65$ billion emergency intervention by the UK central bank. ${ }^{1}$ Even more recently, Silicon Valley Bank and First Republic Bank collapsed as a result of increased interest rates and the subsequent decline in value of long-term bonds and mortgages. ${ }^{2,3}$

If zero-coupon bonds of all maturities were to exist, any deterministic future cash flow can be replicated by these bonds (which is called a "dedication" strategy), and the problem becomes trivial, at least theoretically. However, in practice dedication is infeasible due to market incompleteness: there are fewer bonds available for trade than the number of payment dates of the liability, or the long-term liability could have a longer maturity than the government bond with longest maturity. Thus, in general, one can only hope to hedge against interest rate risk approximately. The question of fundamental practical importance is how to achieve this goal given the set of bonds available for trade.

In this article, we propose a new method to construct a hedging portfolio that maximizes equity (asset minus liability) under the most adversarial interest rate shock. This so-called maxmin problem originates in the work of Fisher and Weil (1971), who show that a portfolio that matches value and duration (weighted average time to payment) is maxmin against parallel shocks to the forward rate. In this paper and subsequent work, the liability is assumed to be a zero-coupon bond and no-shortsale constraints are imposed (or implicitly assumed not to bind). These restrictions are undesirable in practice because most liabilities pay out over time and shortsales are essential when liabilities have very long maturities (like pensions). Furthermore, there is no systematic analysis of the existence, uniqueness, and optimality of the solution method as well as explicit or tight error estimates.

Our approach overcomes these shortcomings using techniques from functional and numerical analysis. First we argue that the most general formulation of the maxmin problem is intractable because the objective function is not convex and the space has infinite dimension. To make the problem manageable, we approximate the objective function using the Gateaux differential with respect to basis functions that approximate yield curve shifts. This allows us to recast the maxmin problem

[^1]as a saddle point (minmax) problem where the inner maximization is a large linear programming problem and the outer minimization is a small convex programming problem, which is computationally tractable. We prove that a robust immunizing portfolio generically exists (Proposition 3.1) and its solution achieves the smallest error order and maximizes the worst-case equity (Theorem 3.3). This maxmin result is significantly different from the existing literature because both the liability structure and bond portfolio constraint are arbitrary and the guaranteed equity bound is tight. When the majority of forward rate changes are captured by a small number of principal components such as the level of the overall interest rate, we improve this guaranteed equity bound by incorporating moment matching (e.g., duration matching) in the portfolio constraint (Theorem 3.5). We also propose particular basis functions (transformation of Chebyshev polynomials) that are motivated by approximation theory.

An alternative approach to asset-liability management, referred to as classical immunization (see, e.g., Redington (1952)), involves matching the interest rate sensitivity of assets and liabilities. A common measure of interest rate sensitivity is duration, and matching the duration of assets and liabilities makes equity insensitive to small interest rate changes. Although classical immunization is intuitive and elegant, by assumption it only allows for small parallel shifts in the yield curve. Furthermore, when there are multiple bonds, it is not obvious how to construct the portfolio because there are infinitely many linear combinations that achieve the same duration. Extensions such as high-order duration matching (which are designed to allow non-infinitesimal or non-parallel shifts in the yield curve) result in unstable portfolio weights and extreme leverage, leading to poor performance (Mantilla-Garcia et al., 2022). Our approach contains classical immunization and its extensions as a special case by choosing a monomial basis and imposing only a value matching constraint. In simulation, we show that our preferred robust immunization method that combines moment matching and a Chebyshev polynomial basis does not suffer from extreme leverage and significantly outperforms existing methods.

The simulation exercise uses historical yield curve data to evaluate the change in equity resulting from instantaneous yield curve shocks. A hedging method's success is measured by its ability to minimize these equity changes. Indeed, we find that robust immunization generates approximation errors that are an order of magnitude smaller than the existing approaches and has lower downside risk, in line with our maxmin result. This numerical experiment has a static flavor, since we only consider one-time perturbations. In a separate simulation based on
a no-arbitrage term structure model, we consider the dynamic properties of robust immunization, allowing for portfolio rebalancing every three months. Over a 10year period of rebalancing, robust immunization achieves a funding ratio (assets to liabilities) of $99 \%$ in the $1 \%$ worst-case scenario and consistently maintains a funding ratio higher than that of existing methods. Because our approach is model-free, we expect our proposed method to be useful for practitioners in assetliability management. ${ }^{4}$

### 1.1 Related literature

When inputs to a problem such as beliefs, information, or shocks are complicated, it is common to optimize against the worst case scenario, i.e., solve the maxmin problem (Gilboa and Schmeidler, 1989; Bergemann and Morris, 2005; Du, 2018; Brooks and Du , 2021). In the context of asset-liability management, Redington (1952, p. 290) considers the Taylor expansion of assets minus liabilities in response to a small change in the (constant) interest rate and anticipates the importance of convexity to guarantee the portfolio value. Fisher and Weil (1971) formalize this idea and show that if the liability is a zero-coupon bond and a bond portfolio matches the value and duration, then the portfolio value can never fall below liabilities under any parallel shift to the forward rate. Bierwag and Khang (1979) show that when the investor has a fixed budget to invest in bonds, then classical immunization (duration matching) is maxmin in the sense that it maximizes the worst possible rate of return under any parallel shift to the forward rate. Fong and Vasicek (1984) consider any perturbation to the forward curve such that the slope of the forward curve is bounded by some constant and derive a lower bound on the portfolio return over the investment horizon that is proportional to it. The constant of proportionality is a measure of interest rate risk and is called " $M$-squared". Minimization of $M$-squared renders a portfolio that minimizes the likelihood of a deviation from liabilities. Zheng (2007) considers perturbations to the forward rate that are Lipschitz continuous, derives the maximum deviation of the bond value, and applies it to a portfolio choice problem.

Several classical books and papers such as Macaulay (1938), Hicks (1939, pp. 184-188), and Samuelson (1945) discovered that the average time to payment ("duration") of a bond captures the interest rate sensitivity of the bond with re-

[^2]spect to parallel shifts in the yield curve. Redington (1952) suggested matching the duration of the asset and liability ("immunization") to hedge against interest rate risk. Chambers et al. (1988), Nawalkha and Lacey (1988) and Prisman and Shores (1988) use polynomials to approximate the yield curve and discuss immunization using high-order duration measures. Ho (1992) introduced the concept of "key rate duration", which is the bond price sensitivity with respect to local shifts in the yield curve at certain key rates (e.g., 10-year yield). Litterman and Scheinkman (1991) use principal component analysis (PCA) to identify common factors that affect bond returns and find that the three factors called level, slope, and curvature explain a large fraction of the variations in returns. Using these factors, Willner (1996) defines level, slope, and curvature durations and shows how they can be used for asset-liability management. See Sydyak (2016) for a review of this literature. In a recent paper, Onatski and Wang (2021) argue that PCA based on the yield curve is prone to spurious analysis since bond yields are highly persistent. As a result, Crump and Gospodinov (2022) show that PCA tends to favor a much lower dimension of the factor space than the true dimension, which can lead to large costs in bond portfolio management. We further discuss our contribution relative to the literature in Section 3.3.

## 2 Problem statement

### 2.1 Model setup

Time is continuous and denoted by $t \in[0, T]$, where $T>0$ is the planning horizon. There are finitely many bonds indexed by $j=1, \ldots, J$, where $J \geq 2$. The cumulative payout of bond $j$ is denoted by the (weakly) increasing function $F_{j}:[0, T] \rightarrow \mathbb{R}_{+}$. For instance, if bond $j$ is a zero-coupon bond with face value normalized to 1 and maturity $t_{j}$, then

$$
F_{j}(t)= \begin{cases}0 & \text { if } 0 \leq t<t_{j}  \tag{2.1}\\ 1 & \text { if } t_{j} \leq t \leq T\end{cases}
$$

Similarly, if bond $j$ continuously pays out coupons at rate $c_{j}$ and has zero face value, then $F_{j}(t)=c_{j} t$ for $0 \leq t \leq T$.

The fund manager seeks to immunize future cash flows against interest rate risk by forming a portfolio of bonds $j=1, \ldots, J$. Let $F:[0, T] \rightarrow \mathbb{R}_{+}$be the cumulative cash flow to be immunized and $y:[0, T] \rightarrow \mathbb{R}$ be the yield curve, which
the fund manager takes as given. The present discounted value of cash flows is given by the Riemann-Stieltjes integral

$$
\begin{equation*}
\int_{0}^{T} \mathrm{e}^{-t y(t)} \mathrm{d} F(t) \tag{2.2}
\end{equation*}
$$

Because the expression $t y(t)$ appears elsewhere, it is convenient to introduce the notation $x(t):=t y(t)$. Note that by the definition of the instantaneous forward rate, we have

$$
\begin{equation*}
x(t)=\int_{0}^{t} f(u) \mathrm{d} u \tag{2.3}
\end{equation*}
$$

where $f(u)$ is the instantaneous forward rate at term $u$. Because $x$ is the integral of forward rates, we refer to it as the cumulative discount rate. Using $x$, we can rewrite the present discounted value of cash flows (2.2) as

$$
\begin{equation*}
P(x):=\int_{0}^{T} \mathrm{e}^{-x(t)} \mathrm{d} F(t) \tag{2.4}
\end{equation*}
$$

which is a functional of $x$. The price $P_{j}(x)$ of bond $j$ can be defined analogously. The fund manager's problem is to approximate $P(x)$ using a linear combination of bonds $\left\{P_{j}(x)\right\}_{j=1}^{J}$ in a way such that the approximation is robust against perturbations to the yield curve $y$ (and hence the cumulative discount rate $x$ ).

### 2.2 Problem

We now formulate the fund manager's problem. Let $\mathcal{Z} \subset \mathbb{R}^{J}$ and $\mathcal{H}$ be the sets of admissible portfolios and perturbations to the cumulative discount rate, respectively. We consider the following maxmin problem:

$$
\begin{equation*}
\sup _{z \in \mathcal{Z}} \inf _{h \in \mathcal{H}}\left[\sum_{j=1}^{J} z_{j} P_{j}(x+h)-P(x+h)\right] . \tag{2.5}
\end{equation*}
$$

Here, the objective function $\sum_{j=1}^{J} z_{j} P_{j}(x+h)-P(x+h)$ represents the difference between assets and liabilities, or "equity". The interpretation of the maxmin problem (2.5) is as follows. Given the portfolio $z \in \mathcal{Z}$, nature chooses the most adversarial perturbation $h \in \mathcal{H}$ to minimize equity. The fund manager chooses the portfolio $z$ that guarantees the highest equity under the worst possible perturbation.

### 2.3 Assumptions

The maxmin problem (2.5) is not tractable because we have not yet specified the admissible sets $\mathcal{Z}, \mathcal{H}$ and the objective function is nonlinear (not even convex) in $h$. We thus impose several assumptions to make progress.

Assumption 1 (Discrete payouts). The bonds and liability pay out on finitely many dates, whose union is denoted by $\left\{t_{n}\right\}_{n=1}^{N} \subset(0, T]$.

Assumption 1 always holds in practice. Under this assumption, each $F_{j}$ is a step function with discontinuities at points contained in $\left\{t_{n}\right\}_{n=1}^{N}$, and integrals of the form (2.4) reduce to summations.

Assumption 2 (Portfolio constraint). The set of admissible portfolios $\mathcal{Z} \subset \mathbb{R}^{J}$ is nonempty and closed. Furthermore, all $z \in \mathcal{Z}$ satisfy value matching:

$$
\begin{equation*}
P(x)=\sum_{j=1}^{J} z_{j} P_{j}(x) . \tag{2.6}
\end{equation*}
$$

Value matching (2.6) is merely a normalization to make the initial equity (assets minus liabilities) equal to 0 . This assumption is common in the immunization literature (see, for example, Bierwag and Khang (1979)).

We now specify the space of cumulative discount rates and their perturbations. Let $C^{r}[0, T]$ be the vector space of $r$-times continuously differentiable functions on $[0, T]$, with the convention that $C^{0}[0, T]=C[0, T]$ denote the space of continuous functions. We let the space of forward rates be the Banach space of continuous functions $C[0, T]$ endowed with the supremum norm denoted by $\|\cdot\|_{\infty} .{ }^{5}$ Since by definition the cumulative discount rate is the integral of the forward rate, if $f$ is continuous, then $x:[0, T] \rightarrow \mathbb{R}$ defined by (2.3) is continuously differentiable with $x(0)=0$. We define the space of cumulative discount rates by

$$
\begin{equation*}
\mathcal{X}=\left\{x \in C^{1}[0, T]: x(0)=0\right\} . \tag{2.7}
\end{equation*}
$$

Lemma A. 1 in the Appendix shows that $\mathcal{X}$ is a Banach space endowed with the norm $\|x\|_{\mathcal{X}}=\sup _{t \in[0, T]}\left|x^{\prime}(t)\right|$. The set of admissible perturbations is a subset $\mathcal{H} \subset \mathcal{X}$. The next assumption allows us to approximate any element $x \in \mathcal{X}$.

Assumption 3. There exists a countable basis $\left\{h_{i}\right\}_{i=1}^{\infty}$ of $\mathcal{X}$ such that for each $1 \leq I \leq N$, the $I \times N$ matrices $H=\left(h_{i}\left(t_{n}\right)\right)$ and $G=\left(h_{i}^{\prime}\left(t_{n}\right)\right)$ have full row rank.

[^3]We refer to each $h_{i}$ as a basis function. Assumption 3 says that the basis functions $\left\{h_{i}\right\}$ and their derivatives $\left\{h_{i}^{\prime}\right\}$ are linearly independent when evaluated on the payout dates. We impose this assumption to avoid portfolio indeterminacy. In practice, we can always ensure that $H$ and $G$ have full row rank by removing certain basis functions if necessary. A typical example satisfying Assumption 3 is to let $h_{i}$ be a polynomial of degree $i$ with $h_{i}(0)=0$ (Lemma A.2).

## 3 Robust asset-liability management

In this section we solve the maxmin problem (2.5) in the limit when the admissible set of perturbations $\mathcal{H}$ shrinks to $\{0\}$. In practice, the resulting portfolio solution is expected to provide a good hedge against the worst-case interest rate shock when the change in interest rates is small.

### 3.1 Robust immunization

As the set of cumulative discount rates $\mathcal{X}$ forms an infinite-dimensional vector space, we employ tools from functional analysis to analyze how prices change in response to perturbations in the discount rate, denoted by $h \in \mathcal{X}$. These perturbations can take various forms, such as a parallel shift, characterized by a constant function $h(t) \equiv c \in \mathbb{R}$, or a linear shift represented by $h(t)=c t$. We assess the price change following an arbitrary shift in the cumulative discount rate by using the Gateaux differential of $P(x):^{6}$

$$
\begin{equation*}
\delta P(x ; h):=\lim _{\alpha \rightarrow 0} \frac{1}{\alpha}(P(x+\alpha h)-P(x))=-\int_{0}^{T} \mathrm{e}^{-x(t)} h(t) \mathrm{d} F(t) . \tag{3.1}
\end{equation*}
$$

Remark 1. The operator $h \mapsto \delta P(x ; h)$ defined by (3.1) is a bounded linear operator from $\mathcal{X}$ to $\mathbb{R}$ (Lemma A.3), which is called the Fréchet derivative and denoted by $P^{\prime}(x)$. Thus by definition $P^{\prime}(x) h=\delta P(x ; h)$. In broad terms, $P^{\prime}(x) h$ quantifies the first-order impact on price change when the cumulative discount rate curve is perturbed by $h$.

Our approach to constructing a maxmin solution is based on assessing the sensitivity of assets and liabilities to perturbations in specific directions $h$. Specifically, given the basis functions $\left\{h_{i}\right\}_{i=1}^{I}$ and bonds $j=1, \ldots, J$, we define the

[^4]sensitivity matrix $A=\left(a_{i j}\right) \in \mathbb{R}^{I \times J}$, where each element $a_{i j}$ represents the sensitivity of bond $j$ (with $F=F_{j}$ ) to a perturbation evaluated at $h=h_{i}$. The exact expression for $a_{i j}$ is given by
\[

$$
\begin{equation*}
a_{i j}:=-\frac{P_{j}^{\prime}(x) h_{i}}{P(x)}=-\frac{\delta P_{j}\left(x ; h_{i}\right)}{P(x)}=\frac{1}{P(x)} \int_{0}^{T} \mathrm{e}^{-x(t)} h_{i}(t) \mathrm{d} F_{j}(t) . \tag{3.2}
\end{equation*}
$$

\]

Division by $P(x)$ is merely a normalization to make $a_{i j}$ dimensionless. Similarly, we define the sensitivity vector $b=\left(b_{i}\right) \in \mathbb{R}^{I}$ of liabilities by

$$
\begin{equation*}
b_{i}:=-\frac{P^{\prime}(x) h_{i}}{P(x)}=-\frac{\delta P\left(x ; h_{i}\right)}{P(x)}=\frac{1}{P(x)} \int_{0}^{T} \mathrm{e}^{-x(t)} h_{i}(t) \mathrm{d} F(t) . \tag{3.3}
\end{equation*}
$$

If $h \in \operatorname{span}\left\{h_{i}\right\}_{i=1}^{I}$, so $h=\sum_{i=1}^{I} w_{i} h_{i}$ for some $w \in \mathbb{R}^{I}$, then under Assumption 2 we obtain

$$
\lim _{\alpha \rightarrow 0} \frac{1}{\alpha P(x)}\left[\sum_{j=1}^{J} z_{j} P_{j}(x+\alpha h)-P(x+\alpha h)\right]=-\langle w, A z-b\rangle,
$$

where $\langle\cdot, \cdot\rangle$ denotes the inner product. Hence, the change in equity following an infinitesimal perturbation in the discount rate is governed by the assets and liabilities' Fréchet derivative. If the portfolio is chosen such that $A z=b$, i.e., the Fréchet derivatives of assets and liabilities are matched, then the worst-case equity is insensitive to small perturbations in the yield curve. We will use this insight to construct the maxmin solution in Theorem 3.3. Before doing so, we present several auxiliary results. In the discussion below, it is convenient to introduce notation for the value matching constraint, which is always assumed to hold (Assumption 2). Specifically, set $h_{0} \equiv 1$ and define $a_{0 j}$ using (3.2). Define the $1 \times J$ vector $a_{0}:=\left(a_{0 j}\right)$ and the $(I+1) \times J$ matrix and $(I+1) \times 1$ vector

$$
A_{+}:=\left[\begin{array}{c}
a_{0}  \tag{3.4}\\
A
\end{array}\right] \quad \text { and } \quad b_{+}:=\left[\begin{array}{l}
1 \\
b
\end{array}\right] .
$$

In what follows, longer proofs are relegated to Appendix B.
Proposition 3.1 (Minmax). Suppose Assumptions 1-3 hold, $I \geq J-1$, and $A_{+}$ in (3.4) has full column rank. Define the $I \times N \operatorname{matrix} G=\left(h_{i}^{\prime}\left(t_{n}\right)\right)$ and the set

$$
\begin{equation*}
\mathcal{W}:=\left\{w \in \mathbb{R}^{I}: G^{\prime} w \in[-1,1]^{N}\right\} . \tag{3.5}
\end{equation*}
$$

Then there exists $\left(z^{*}, w^{*}\right) \in \mathcal{Z} \times \mathcal{W}$ that achieves the minmax value

$$
\begin{equation*}
V_{I}(\mathcal{Z}):=\inf _{z \in \mathcal{Z}} \sup _{w \in \mathcal{W}}\langle w, A z-b\rangle . \tag{3.6}
\end{equation*}
$$

Furthermore, $V_{I}(\mathcal{Z}) \geq 0$, and $z \in \mathcal{Z}$ achieves $V_{I}(\mathcal{Z})=0$ if and only if $A_{+} z=b_{+}$.
The matrix $G=\left(h_{i}^{\prime}\left(t_{n}\right)\right)$ can be thought of as the $I \times N$ matrix of perturbations to forward rates. The set $\mathcal{W}$ in (3.5) thus characterizes the span of perturbations to the forward rate that are bounded in absolute value by one. Proposition 3.1 assumes that $A_{+}$in (3.4) has full column rank. If the cumulative payouts of bonds $\left\{F_{j}\right\}$ and the basis functions $\left\{h_{i}\right\}$ are linearly independent, the matrix $A_{+}$generically has full column rank and therefore a solution $(z, w) \in \mathcal{Z} \times \mathcal{W}$ to the minmax problem (3.6) generically exists. Appendix C makes this statement precise.

The solution $z$ to the minmax problem (3.6) depends on the basis functions $\left\{h_{i}\right\}_{i=1}^{I}$ only through its span and it is immaterial how we parameterize these functions. Although this result is obvious, we note it as a proposition.

Proposition 3.2 (Basis invariance). Let everything be as in Proposition 3.1 and $\mathcal{Z}^{*}$ be the set of solutions $z^{*} \in \mathcal{Z}$ to the minmax problem (3.6). Then $V_{I}(\mathcal{Z})$ and $\mathcal{Z}^{*}$ depend on the basis functions $\left\{h_{i}\right\}_{i=1}^{I}$ only through its span.

For any bond portfolio $z \in \mathcal{Z}$, define the portfolio share $\theta=\left(\theta_{j}\right) \in \mathbb{R}^{J}$ by

$$
\begin{equation*}
\theta_{j}:=z_{j} P_{j}(x) / P(x) . \tag{3.7}
\end{equation*}
$$

Under Assumption 2, the portfolio share $\theta$ satisfies $\sum_{j=1}^{J} \theta_{j}=1$. Therefore the $\ell^{1}$ norm $\|\theta\|_{1}=\sum_{j=1}^{J}\left|\theta_{j}\right|$ satisfies $\|\theta\|_{1}=1$ if and only if $\theta_{j} \geq 0$ for all $j$, and $\|\theta\|_{1}>1$ is equivalent to $\theta_{j}<0$ for some $j$. Thus $\|\theta\|_{1}$ can be interpreted as a measure of leverage, which we refer to as the gross leverage.

To state our main result, we consider the following set of admissible perturbations to the cumulative discount rate for any $\Delta>0$ :

$$
\begin{equation*}
\mathcal{H}_{I}(\Delta):=\left\{h \in \operatorname{span}\left\{h_{i}\right\}_{i=1}^{I}:(\forall n)\left|h^{\prime}\left(t_{n}\right)\right| \leq \Delta\right\} . \tag{3.8}
\end{equation*}
$$

Because $h$ is a perturbation to the cumulative discount rate, which is the integral of the forward rate, choosing $h \in \mathcal{H}_{I}(\Delta)$ amounts to allowing the forward rates to change by at most $\Delta$ while spanned by the first $I$ basis functions. The following theorem is our main theoretical result.

Theorem 3.3 (Robust immunization). Let everything be as in Proposition 3.1 and $\mathcal{H}_{I}(\Delta)$ be as in (3.8). Then the guaranteed equity satisfies

$$
\begin{equation*}
\lim _{\Delta \downarrow 0} \frac{1}{\Delta} \sup _{z \in \mathcal{Z}} \inf _{h \in \mathcal{H}_{I}(\Delta)}\left[\sum_{j=1}^{J} z_{j} P_{j}(x+h)-P(x+h)\right]=-P(x) V_{I}(\mathcal{Z}) . \tag{3.9}
\end{equation*}
$$

Letting $z^{*} \in \mathcal{Z}$ be the solution to the minmax problem (3.6) and $\theta=\left(\theta_{j}\right) \in \mathbb{R}^{J}$ be the corresponding portfolio share defined by (3.7), then

$$
\begin{equation*}
\sup _{h \in \mathcal{H}_{I}(\Delta)}\left|P(x+h)-\sum_{j=1}^{J} z_{j}^{*} P_{j}(x+h)\right| \leq \Delta P(x)\left(V_{I}(\mathcal{Z})+\frac{1}{4} \Delta T^{2} \mathrm{e}^{\Delta T}\left(1+\|\theta\|_{1}\right)\right) . \tag{3.10}
\end{equation*}
$$

Theorem 3.3 has several implications. First, (3.9) shows that, to the first order, the guaranteed equity is exactly $-\Delta P(x) V_{I}(\mathcal{Z})$ when forward rates (hence yields) are perturbed by at most $\Delta$ within the span of the basis functions. The minmax value $V_{I}(\mathcal{Z})$ has a natural interpretation and is the answer to the following question: "if forward rates change by at most one percentage point, what is the largest percentage point decline in the portfolio value?" The maxmin formula (3.9) provides an exact characterization of the worst-case outcome, and the number $V_{I}(\mathcal{Z})$ can be solved as the minmax value (3.6).

Second, the error estimate (3.10) shows that the solution $z^{*} \in \mathcal{Z}$ to the minmax problem (3.6) achieves the lower bound in (3.9), to the first order. In this sense $z^{*}$ is an optimal portfolio, which we refer to as the robust immunizing portfolio. Clearly, this immunizing portfolio is independent of $\Delta>0$ as the minmax problem (3.6) does not involve $\Delta$. In addition, the minmax value (3.6) satisfies the following comparative statics.

Proposition 3.4 (Monotonicity of minmax value). Let everything be as in Proposition 3.1. If $I<I^{\prime}$ and $\mathcal{Z} \subset \mathcal{Z}^{\prime}$, then $V_{I}(\mathcal{Z}) \leq V_{I^{\prime}}(\mathcal{Z})$ and $V_{I}(\mathcal{Z}) \geq V_{I}\left(\mathcal{Z}^{\prime}\right)$.

The result $V_{I}(\mathcal{Z}) \leq V_{I^{\prime}}(\mathcal{Z})$ is obvious because the more basis functions we use, the more freedom nature has to select adversarial perturbations. The result $V_{I}(\mathcal{Z}) \geq V_{I}\left(\mathcal{Z}^{\prime}\right)$ is also obvious because the larger the set of admissible portfolios is, the more freedom the fund manager has to select portfolios.

### 3.2 Robust immunization with principal components

So far we have put no structure on the basis functions $\left\{h_{i}\right\}_{i=1}^{I}$ beyond Assumption 3. The set of admissible perturbations (3.8) depends only on span $\left\{h_{i}\right\}_{i=1}^{I}$ and the
particular order or parameterization does not matter. However, in practice there could be some factor structure in the forward rate. For instance, a typical shift to the forward curve might be decomposed into the sum of a parallel shift and a nonparallel shift of a smaller size. In this section we formalize this idea and extend Theorem 3.3 to a setting where the perturbation in a particular direction (principal component) could be larger.

For any $\Delta_{1}, \Delta_{2}>0$, consider the following admissible set of perturbations:

$$
\begin{align*}
& \mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right) \\
& \quad=\left\{h \in \operatorname{span}\left\{h_{i}\right\}_{i=1}^{I}:(\exists \alpha)(\forall n)\left|\alpha h_{1}^{\prime}\left(t_{n}\right)\right| \leq \Delta_{1},\left|h^{\prime}\left(t_{n}\right)-\alpha h_{1}^{\prime}\left(t_{n}\right)\right| \leq \Delta_{2}\right\} . \tag{3.11}
\end{align*}
$$

Choosing $h \in \mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right)$ amounts to perturbing the forward rate in the direction spanned by the first component $\left(h_{1}^{\prime}\right)$ by a magnitude at most $\Delta_{1}$, and then perturbing in an arbitrary direction spanned by the first $I$ basis functions by a magnitude at most $\Delta_{2}$. Thus setting $\Delta_{1} \gg \Delta_{2}$ captures the idea that $h_{1}$ is the first principal component. In this setting, we can generalize Theorem 3.3 as follows.

Theorem 3.5 (Robust immunization with principal components). Let everything be as in Proposition 3.1 and suppose the set

$$
\begin{equation*}
\mathcal{Z}_{1}:=\left\{z \in \mathcal{Z}: \sum_{j=1}^{J} a_{1 j} z_{j}=b_{1}\right\} \tag{3.12}
\end{equation*}
$$

is nonempty, where $a_{1 j}$ and $b_{1}$ are defined by (3.2) and (3.3) with $i=1$. Let $\mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right)$ be as in (3.11). Then the guaranteed equity satisfies

$$
\begin{equation*}
\lim \frac{1}{\Delta_{2}} \sup _{z \in \mathcal{Z}} \inf _{h \in \mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right)}\left[\sum_{j=1}^{J} z_{j} P_{j}(x+h)-P(x+h)\right]=-P(x) V_{I}\left(\mathcal{Z}_{1}\right) \tag{3.13}
\end{equation*}
$$

where the limit is taken over $\Delta_{1}, \Delta_{2} \rightarrow 0, \Delta_{1} / \Delta_{2} \rightarrow \infty$, and $\Delta_{1}^{2} / \Delta_{2} \rightarrow 0$. Letting $z^{*} \in \mathcal{Z}_{1}$ be the solution to the minmax problem (3.6) with portfolio constraint $\mathcal{Z}_{1}$, we have

$$
\begin{equation*}
\sup _{h \in \mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right)}\left|P(x+h)-\sum_{j=1}^{J} z_{j}^{*} P_{j}(x+h)\right| \leq \Delta_{2} P(x)\left(V_{I}\left(\mathcal{Z}_{1}\right)+O\left(\Delta_{2}+\Delta_{1}^{2} / \Delta_{2}\right)\right) . \tag{3.14}
\end{equation*}
$$

The value added of Theorem 3.5 relative to Theorem 3.3 can be explained as
follows. Comparing to (3.11) to (3.8) and applying the triangle inequality

$$
\left|h^{\prime}(t)\right| \leq\left|\alpha h_{1}^{\prime}(t)\right|+\left|h^{\prime}(t)-\alpha h_{1}^{\prime}(t)\right|
$$

we obtain $\mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right) \subset \mathcal{H}_{I}\left(\Delta_{1}+\Delta_{2}\right)$. Therefore to first-order, the maximum portfolio return loss can be bounded as

$$
\underbrace{\Delta_{2} V_{I}\left(\mathcal{Z}_{1}\right)}_{\text {Theorem 3.5 }} \leq \underbrace{\left(\Delta_{1}+\Delta_{2}\right) V_{I}(\mathcal{Z})}_{\text {Theorem 3.3 }}
$$

Thus if $\Delta_{1} \gg \Delta_{2}$ in typical situations (see Figure 2), then imposing the constraint $\mathcal{Z}_{1}$ in (3.12) improves the performance. ${ }^{7}$

Remark 2. Theorem 3.5 can be further generalized if we allow larger perturbations spanned by the first few basis functions. For instance, if we use the first two basis functions, we can define $\mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}, \Delta_{3}\right)$ analogously to (3.11) by incorporating the constraints $\left|\alpha_{i} h_{i}^{\prime}\left(t_{n}\right)\right| \leq \Delta_{i}$ for $i=1,2$ and $\left|h^{\prime}\left(t_{n}\right)-\alpha_{1} h_{1}^{\prime}\left(t_{n}\right)-\alpha_{2} h_{2}^{\prime}\left(t_{n}\right)\right| \leq \Delta_{3}$. The portfolio constraint (3.12) then becomes

$$
\begin{equation*}
\mathcal{Z}_{2}:=\left\{z \in \mathcal{Z}: \sum_{j=1}^{J} a_{i j} z_{j}=b_{i} \text { for } i=1,2\right\}, \tag{3.15}
\end{equation*}
$$

and the maxmin formula (3.13) involves $V_{I}\left(\mathcal{Z}_{2}\right)$.

### 3.3 Relation to existing literature

In this section we discuss in some detail how Theorem 3.3 is related to the existing literature. The following corollary shows that when $I=J-1$ and there is no portfolio constraint beyond value matching, the immunizing portfolio can be solved explicitly.

Corollary 3.6 (Robust immunization with $I=J-1$ ). Let everything be as in Proposition 3.1 and suppose that the only portfolio constraint is value matching (2.6), so the set of admissible portfolios is

$$
\begin{equation*}
\mathcal{Z}_{0}:=\left\{z \in \mathbb{R}^{J}: P(x)=\sum_{j=1}^{J} z_{j} P_{j}(x)\right\} . \tag{3.16}
\end{equation*}
$$

[^5]If $I=J-1$ and the square matrix $A_{+}$in (3.4) is invertible, then the unique solution to (3.6) is $z^{*}=A_{+}^{-1} b_{+}$, with $V_{I}(\mathcal{Z})=0$.

Proof. Immediate from the proof of Proposition 3.1.
Remark 3. The special case of Corollary 3.6 with $I=J-1=1$ and $h_{1}(t)=t$ reduces to classical immunization that matches the bond value and duration. To see this, recall that the duration of the cash flow $F$ is defined by the weighted average time to payment

$$
D=\frac{\int_{0}^{T} t \mathrm{e}^{-t y(t)} \mathrm{d} F(t)}{\int_{0}^{T} \mathrm{e}^{-t y(t)} \mathrm{d} F(t)}
$$

Using the definition $x(t)=t y(t)$ and (3.1), the duration can be rewritten as

$$
D=\frac{\int_{0}^{T} t \mathrm{e}^{-x(t)} \mathrm{d} F(t)}{\int_{0}^{T} \mathrm{e}^{-x(t)} \mathrm{d} F(t)}=-\frac{P^{\prime}(x) h_{1}}{P(x)}=b_{1},
$$

where $h_{1}(t)=t$ and we have used (3.3). A similar calculation implies that the duration of the immunizing portfolio is

$$
-\frac{\sum_{j=1}^{J} z_{j} P_{j}^{\prime}(x) h_{1}}{\sum_{j=1}^{J} z_{j} P_{j}(x)}=-\frac{\sum_{j=1}^{J} z_{j} P_{j}^{\prime}(x) h_{1}}{P(x)}=\sum_{j=1}^{J} a_{1 j} z_{j}
$$

using value matching (2.6) and (3.2). Therefore if $z=A_{+}^{-1} b_{+}$, so $A_{+} z=b_{+}$, the duration is matched. By the same argument, setting $I=J-1$ and $h_{i}(t)=t^{i}$ reduces to high-order duration matching ( $I=J-1=2$ is convexity matching).

Remark 4. Proposition 3.4 explains why high-order duration matching ( $I=J-1$, no portfolio constraint, and $h_{i}(t)=t^{i}$ ) does not necessarily have good performance (Mantilla-Garcia et al., 2022). When $I=J-1$, as we increase $I$, both the number of basis functions $I$ and the set of admissible portfolios $\mathcal{Z}$ expand. Because increasing $I$ makes $V_{I}(\mathcal{Z})$ larger but expanding $\mathcal{Z}$ makes it smaller, the combined effect could go either way.

In addition to the setting in Corollary 3.6, if the liability pays out on a single date and the immunizing portfolio does not involve shortsales, we can obtain the following global result.

Proposition 3.7 (Guaranteed funding). Let everything be as in Corollary 3.6 and suppose that the liability pays out on a single date. If $z^{*}=A_{+}^{-1} b_{+} \geq 0$, then for
all $h \in \operatorname{span}\left\{h_{i}\right\}_{i=1}^{I}$ we have

$$
\begin{equation*}
\sum_{j=1}^{J} z_{j}^{*} P_{j}(x+h) \geq P(x+h) \tag{3.17}
\end{equation*}
$$

Remark 5. Our maxmin result (Theorems 3.3 and 3.5) is quite different from the existing literature such as Fisher and Weil (1971) and Bierwag and Khang (1979). To the best of our knowledge, in this literature it is always assumed that the liability pays out on a single date and the portfolio does not involve shortsales $(z \geq$ 0 ) yet this constraint does not bind. Under these assumptions, Proposition 3.7 shows that the immunizing portfolio always funds the liability, which generalizes the result of Fisher and Weil (1971) (who proved (3.17) for $I=J-1=1$ and $\left.h_{1}(t)=t\right)$. However, this result is quite restrictive because liabilities could be paid out over time and shortsales are essential when the maturity of the liability is very long (such as pensions). Our maxmin result (3.9) accommodates arbitrary liability structures and portfolio constraints.

### 3.4 Implementation

To implement robust immunization, we need to choose the basis functions $\left\{h_{i}\right\}_{i=1}^{I}$. For each $i$, it is natural to choose $h_{i}$ such that $h_{i}$ is a polynomial of degree $i$ with $h_{i}(0)=0$, for Assumption 3 then holds (Lemma A.2). By basis invariance (Proposition 3.2), any choice of such a basis will result in the same immunizing portfolio.

However, we suggest using Chebyshev polynomials because they enjoy good approximation properties (Trefethen, 2019, Ch. 2-4). To be more specific, let $T_{n}$ : $[-1,1] \rightarrow \mathbb{R}$ be the $n$-degree Chebyshev polynomial defined by $T_{n}(\cos \theta)=\cos n \theta$ and setting $x=\cos \theta$. We map $[0, T]$ to $[-1,1]$ using $t \mapsto x=2 t / T-1$, and define $g_{i}:[0, \infty) \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
g_{i}(t)=T_{i-1}(2 t / T-1) \tag{3.18}
\end{equation*}
$$

so that we can allow any (continuous) perturbations to the forward rate for $t \in$ $[0, T]$. Then our basis functions for perturbing the cumulative discount rate (the integral of the forward rate) can be defined by

$$
\begin{equation*}
h_{i}(t)=\int_{0}^{t} g_{i}(u) \mathrm{d} u \tag{3.19}
\end{equation*}
$$

for each $i$. The following lemma provides explicit formulas for the basis functions

Lemma 3.8 (Chebyshev basis for forward rate). Let $T_{n}$ be the $n$-degree Chebyshev polynomial defined by $T_{n}(\cos \theta)=\cos n \theta$ and setting $x=\cos \theta$. For each $i$, the basis function $h_{i}$ in (3.19) can be expressed as

$$
\begin{align*}
& h_{1}(t)=t,  \tag{3.20a}\\
& h_{2}(t)=\frac{1}{4} T\left((2 t / T-1)^{2}-1\right), \tag{3.20b}
\end{align*}
$$

and for $i \geq 3$,

$$
\begin{equation*}
h_{i}(t)=\frac{1}{4} T\left(\frac{T_{i}(2 t / T-1)}{i}-\frac{T_{i-2}(2 t / T-1)}{i-2}+\frac{2(-1)^{i}}{i(i-2)}\right) . \tag{3.20c}
\end{equation*}
$$

Figure 1a shows the graphs of the first few basis functions (3.20) for $T=50$ years. Figure 1b shows the graphs of $g_{i}=h_{i}^{\prime}$ in (3.18), which are the rows of the matrix $G$ in Proposition 3.1.


Figure 1: Basis functions of robust immunization.

We now describe the algorithm to implement robust immunization in practice. Although the underlying theory (which heavily relies on functional and numerical analysis) may not be familiar to practitioners, the implementation only requires little more than basic linear algebra and linear programming.

## Robust Immunization.

(i) Let $\mathbf{t}=\left(t_{1}, \ldots, t_{N}\right)$ be the $1 \times N$ vector of asset/liability payout dates and $T=t_{N}$ be the planning horizon. Let $\mathbf{y}=\left(y_{1}, \ldots, y_{N}\right)$ be the $1 \times N$ vector of yields, $\mathbf{f}=\left(f_{1}, \ldots, f_{N}\right)$ be the $1 \times N$ vector of liabilities, and $\mathbf{F}=\left(f_{j n}\right)$ be the $J \times N$ matrix of bond payouts.
(ii) Let $I \geq J-1$, define the basis functions by (3.20), evaluate at each $t_{n}$, and construct the $I \times N$ matrix of basis functions $\mathbf{H}=\left(h_{i}\left(t_{n}\right)\right)$ and their derivative $\mathbf{G}=\left(h_{i}^{\prime}\left(t_{n}\right)\right)=\left(g_{i}\left(t_{n}\right)\right)$. Define the $1 \times N$ vector of zero-coupon bond prices $\mathbf{p}=\exp (-\mathbf{y} \odot \mathbf{t})$, where $\odot$ denotes entry-wise multiplication (Hadamard product).
(iii) Define the $I \times J$ matrix $A, I \times 1$ vector $b$, and $1 \times J$ vector $a_{0}$ by

$$
A:=\left(\mathbf{H} \operatorname{diag}(\mathbf{p}) \mathbf{F}^{\prime}\right) /\left(\mathbf{p f}^{\prime}\right), \quad b:=\mathbf{H} \operatorname{diag}(\mathbf{p}) \mathbf{f}^{\prime} /\left(\mathbf{p f}^{\prime}\right), \quad a_{0}:=\mathbf{p F}^{\prime} /\left(\mathbf{p f}^{\prime}\right),
$$

where $\operatorname{diag}(\mathbf{p})$ denotes the diagonal matrix with diagonal entries given by p. Define the $(I+1) \times J$ matrix $A_{+}$and $(I+1) \times 1$ vector $b_{+}$by

$$
A_{+}:=\left[\begin{array}{l}
a_{0} \\
A
\end{array}\right] \quad \text { and } \quad b_{+}:=\left[\begin{array}{l}
1 \\
b
\end{array}\right] .
$$

(iv) If $I=J-1$ and there are no portfolio constraints, calculate the immunizing portfolio as $z^{*}=A_{+}^{-1} b_{+}$. Otherwise, solve the minmax problem (3.6).

Note that the inner maximization in (3.6) is a linear programming problem with $I$ variables and $2 N$ inequality constraints, which is straightforward to solve numerically even when $N$ is large (a few hundred in typical applications). The outer minimization is a convex minimization problem with $J$ variables, which is also straightforward to solve numerically.

## 4 Evaluation: static hedging

In this section we evaluate the performance of robust immunization and other existing methods using a numerical experiment in a static setting.

### 4.1 Experimental design

Data and yield curve model We obtain daily U.S. Treasury nominal yield curve data from November 25, 1985 to September 2022 from the Federal Reserve. ${ }^{8}$ We denote the days by $s=1, \ldots, S$, where $S=9,201$ is the sample size. These daily yield curves are estimated using the methodology of Gürkaynak et al. (2007), who assume that the instantaneous forward rate at term $t$ is specified by the Svensson (1994) model

$$
\begin{equation*}
f(t)=\beta_{0}+\beta_{1} \exp \left(-t / \tau_{1}\right)+\beta_{2}\left(t / \tau_{1}\right) \exp \left(-t / \tau_{1}\right)+\beta_{3}\left(t / \tau_{2}\right) \exp \left(-t / \tau_{2}\right) \tag{4.1}
\end{equation*}
$$

where $\beta_{0}, \beta_{1}, \beta_{2}, \beta_{3} \in \mathbb{R}$ and $\tau_{1}, \tau_{2}>0$ are parameters. The functional form (4.1) allows for two humps in the forward curve that are governed by the parameters $\tau_{1}$ and $\tau_{2}$. Integrating the forward rate in (4.1), we obtain the cumulative discount rate

$$
x(t)=\beta_{0} t-\beta_{1} \tau_{1} \exp \left(-t / \tau_{1}\right)-\beta_{2}\left(t+\tau_{1}\right) \exp \left(-t / \tau_{1}\right)-\beta_{3}\left(t+\tau_{2}\right) \exp \left(-t / \tau_{2}\right)
$$

Note that the parameters in (4.1) change over time, but we suppress the time subscript $s$ for notional clarity. Our data set includes the estimated parameters $\left(\beta_{0}, \beta_{1}, \beta_{2}, \beta_{3}, \tau_{1}, \tau_{2}\right)$ for each day, with which we can evaluate the forward curve (and hence the yield and cumulative discount curves) at arbitrary term $t \geq 0$.

Remark 6. The estimated parameters of Gürkaynak et al. (2007) go back all the way to 1961, but we only use their data beyond $11 / 25 / 1985$ when bonds with a maturity of 30 years were introduced in the market. The authors caution against extrapolation of the forward rate beyond the maximum available bond maturity. Anticipating our empirical application, we need to obtain forward rates with a maturity up to 50 years. Since extrapolation is still necessary in this case, we extrapolate the forward rate by a constant beyond the 30 -year maturity. This approach is motivated by no-arbitrage arguments which stipulate that the long term forward rate is constant (Dybvig et al., 1996). In Appendix E.1, we show how the constant forward rate assumption affects our estimate of the yield curve.

Approximating forward rate changes by basis functions Our theory is based on the assumption that changes in the forward rate can be approximated by the basis functions. To evaluate this assumption, we regress the $d$-day ahead

[^6]forward rate changes on the basis functions $g_{i}$ in (3.18) and calculate a goodness-of-fit measure denoted by $R^{2}$ (see Appendix E. 2 for details).

The left panel of Figure 2 shows this goodness-of-fit measure $R^{2}$ for various horizons $d$. The goodness-of-fit seems to be independent of $d$ except when $I=1$. The first basis function (constant) explains between 50 and $65 \%$ of variations in the forward rate changes, and the first two basis functions (constant and linear) explain about $80 \%$. This result shows that it can be important to account for principal components in constructing the robust immunization portfolio, as in Theorem 3.5. The right panel shows the unexplained component $1-R^{2}$ as we include more basis functions. We can see that setting $I=10$ captures about $99.9 \%\left(1-R^{2}<10^{-3}\right)$ of variations in the forward rate changes.


Figure 2: Goodness-of-fit of forward rate change approximation.
Note: The left panel shows $R^{2}$ for each $d$-day ahead change in the forward rate using the basis functions $\left\{g_{i}\right\}_{i=1}^{I}$ in (3.18) as regressors. The right panel shows the combined $1-R^{2}$ as we increase the number of basis functions $I$. See Appendix E. 2 for details.

Cash flow and immunization methods We now turn to the immunization design. We suppose that the future cash flows of the liability are equal to $1 /(12 T)$ every month for $T=50$ years (so the cumulative cash flow is normalized to 1 ), and the bonds available for trade are zero-coupon bonds with face value 1 and years to maturity being a subset of $\{1,5,10,20,30\}$. We intentionally choose a long maturity of 50 years for the cash flows because it is of interest to study how the yield curve at the long end affects the performance of the immunization methods.

We consider three immunization methods. The first method is high-order duration matching (HD) explained in Remark 3, which is a special case of robust immunization by setting $I=J-1$ and $h_{i}(t)=t^{i}$. By basis invariance (Proposition 3.2 ), we can choose any polynomial basis, so we use the Chebyshev functions in

Lemma 3.8 with $T=50$. The second method is key rate duration matching (KRD) proposed by Ho (1992) and explained in Appendix E.3. In short, this method is designed to match the liability and portfolio sensitivity to interest rate changes at pre-specified maturities. The third method is our proposed robust immunization method (RI) with the Chebyshev basis for the forward rate in Lemma 3.8. Motivated by the right panel of Figure 2, we set the number of basis functions to $I=10$. For the portfolio constraint, motivated by Theorem 3.5 and the left panel of Figure 2, we consider value matching only ( $\mathcal{Z}_{0}$ in (3.16) ), value- and duration matching ( $\mathcal{Z}_{1}$ in (3.12)) and value-, duration- and convexity matching. We refer to these methods as $\operatorname{RI}(0), \operatorname{RI}(1)$ and $\operatorname{RI}(2)$ respectively. ${ }^{9}$

For each method, we consider immunizing the cash flows with $J=2,3,4,5$ bonds. $J=2$ corresponds to using the 1 - and 30 -year zero-coupon bonds, and we add the 5 -, 10 -, and 20 -year bond for $J=3,4,5$, respectively. Note that for HD, $J=2$ is simply classical immunization with duration matching; $J=3$ is duration and convexity matching. For KRD, we use the longer maturity bonds to match the key rates and we use the remaining shortest maturity bond to match value. For example, in case $J=3$, we use the 30 - and 5 -year bond to match the 30 - and 5 -year key rate of liabilities and we use the remaining 1 -year bond to match the value of liabilities.

Return error Suppose that on day $s$, the fund manager immunizes future cash flows with a bond portfolio $z_{s}=\left(z_{s j}\right)$ constructed by the HD, KRD, and RI methods. Motivated by the error estimate (3.10), we evaluate each method using the absolute return error on day $s+d$ defined by

$$
\begin{equation*}
\frac{1}{P\left(x_{s}\right)}\left|P\left(x_{s+d}\right)-\sum_{j=1}^{J} z_{s j} P_{j}\left(x_{s+d}\right)\right|, \tag{4.2}
\end{equation*}
$$

where $x_{s}(t)$ is the cumulative discount rate on day $s$ for term $t$ and we consider the portfolio holding period of $d=1, \ldots, 100$ days. ${ }^{10}$ The performance measure (4.2) can be understood as the return error if after forming the immunizing portfolio on day $s$, the yield curve instantaneously shifts to that of day $s+d$. In this sense the return error (4.2) is a performance measure of static hedging. We address dynamic hedging in Section 5.

[^7]
### 4.2 Results

Figure 3 shows the return error defined by (4.2) averaged over the sample period. The return error worsens with longer portfolio holding periods (d) for all bond quantities and methods because of greater yield curve fluctuations. When there are only two bonds ( $J=2$, Figure 3a), by construction the HD and $\mathrm{RI}(1)$ method agree and they achieve the lowest return error. When there are three bonds $(J=3$, Figure 3b), by construction the HD and $\mathrm{RI}(2)$ methods agree, and they achieve the lowest return error, with $\mathrm{RI}(1)$ close behind. When there are four bonds $(J=4$, Figure 3c), RI(1) clearly outperforms all other methods. Finally, in case of five bonds $(J=5$, Figure $3 d), \operatorname{RI}(1)$ and $\mathrm{RI}(2)$ are the best performing methods with $\operatorname{RI}(2)$ being slightly more accurate over short horizons whereas $\mathrm{RI}(1)$ is more accurate over longer holding periods. Overall, the lowest error is achieved by RI(1) with four bonds. Turning to the existing approaches in the literature, we see that HD does well only for $J \leq 3$, while the performance of KRD is only comparable to robust immunization in case of using five bonds.

Figure 3 presents only average return errors. To evaluate the performance of each method under adversarial circumstances, Table 1 presents the mean, 95and 99 percentiles of the return error for a portfolio holding period of 30 days. According to this table, the performance of the HD method is non-monotonic, which performs best when $J=3$ but deteriorates when $J \geq 4$. The performance of the KRD method monotonically improves with $J$, but it is accurate only when $J=5$. In contrast, $\mathrm{RI}(1)$ and $\mathrm{RI}(2)$ perform well with any number of bonds and their return errors are an order of magnitude lower compared to HD and KRD when $J \geq 4$.

We can summarize the findings in Figure 3 and Table 1 as follows: (i) Regardless of the number of bonds, one of the robust immunization (RI) methods achieves the lowest return error, and generally $\operatorname{RI}(1)$ (matching value and duration) or $\mathrm{RI}(2)$ (matching value, duration, and convexity) is the best. (ii) The performance of the HD method is non-monotonic in $J$, performing best with $J=3$ but poorly with $J \geq 4$. (iii) The performance of KRD is poor for $J \leq 4$ and good for $J=5$.

We next compare the performance of the best specification for each method. For example, we set $J=3$ for HD and $J=5$ for KRD, and we consider $\operatorname{RI}(1)$ for robust immunization with $J=4$ bonds. Figure 4a shows the time series plot of the return error for each immunization method. We see that $\mathrm{RI}(1)$ is dominating the other methods almost uniformly over the entire sample period. Furthermore, KRD


Figure 3: Return error for different holding periods.
Note: The figure presents the return error in portfolio value defined by (4.2) over various holding periods, averaged over the entire sample period. $\mathrm{RI}(0)$ : robust immunization with a value matching; $\mathrm{RI}(1)$ : robust immunization with value and duration matching; $\mathrm{RI}(2)$ : robust immunization with value, duration, and convexity matching; HD: high-order duration matching; KRD: key rate duration matching. The panels in the figure show the error for different number of bonds $J$ used to construct the immunizing portfolio.
meaningfully outperforms HD only before 1990. Figure 4b shows the histogram of the absolute return errors (4.2). We can see that large return errors tend to be less frequent with RI(1). To see this formally, Figure 4 c plots the survival probability of return losses (defined analogously to (4.2) but without taking absolute values) above various thresholds. The fact that RI(1) has lower tail (survival) probability than other methods implies that losses are less likely. Figure 4 d plots the value at risk (VaR) of each method. The value at risk is the quantile of the return distribution and hence the graph plots the size of the return loss corresponding to the specified loss probability. $\mathrm{RI}(1)$ uniformly has the lowest value at risk. These findings are consistent with Theorem 3.5 because the robust immunization method

Table 1: Return error (\%) for 30-day holding period.

| Method: | RI(0) | $\mathrm{RI}(1)$ | $\mathrm{RI}(2)$ | HD | KRD |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Mean |  |  |  |  |  |
| $J=2$ | 0.54 | 0.52 | - | 0.52 | 1.65 |
| $J=3$ | 0.48 | 0.28 | 0.28 | 0.28 | 1.42 |
| $J=4$ | 0.54 | 0.12 | 0.19 | 1.02 | 0.85 |
| $J=5$ | 0.44 | 0.19 | 0.19 | 1.9 | 0.28 |
| $95^{\text {th }}$ percentile |  |  |  |  |  |
| $J=2$ | 1.66 | 1.60 | - | 1.60 | 4.3 |
| $J=3$ | 1.32 | 0.87 | 0.86 | 0.86 | 3.67 |
| $J=4$ | 1.52 | 0.43 | 0.58 | 3.44 | 2.2 |
| $J=5$ | 1.37 | 0.64 | 0.54 | 6.2 | 0.91 |
| 99 ${ }^{\text {th }}$ percentile |  |  |  |  |  |
| $J=2$ | 2.38 | 2.49 | - | 2.49 | 6.85 |
| $J=3$ | 2.19 | 1.84 | 1.77 | 1.77 | 5.99 |
| $J=4$ | 2.53 | 0.85 | 1.17 | 7.62 | 3.59 |
| $J=5$ | 2.42 | 1.07 | 0.91 | 15.15 | 1.49 |

Note: See Figure 3 caption. The best performing method is indicated in bold.
is designed to maximize the return error under the most adversarial perturbation to the cumulative discount rate.

We also test more formally whether the absolute return errors of $\mathrm{RI}(1)$ dominate HD and KRD. To do so, we use the nonparametric sign test which can be used to test whether the median absolute return error is the same for both methods. More details about this test are described in Appendix E.4, where we show that the 30 -day return error for $\mathrm{RI}(1)$ is significantly better than the best performing HD and KRD method.

Leverage To shed light on the observation that the performance in Figure 3 is non-monotonic for HD , Table 2 shows the gross leverage ( $\ell^{1}$ norm) of the portfolio shares $\|\theta\|_{1}=\sum_{j=1}^{J}\left|\theta_{j}\right|$. The leverage for HD portfolios is rather pronounced for $J=4,5$ compared to RI, both in median and in the right tail. Mantilla-Garcia et al. (2022) show that levered portfolios can lead to poor out-of-sample hedging, which can explain the poor performance of HD for $J=4,5$ in Figure 3.


Figure 4: Comparison of best specifications
Note: The figures compare the best specification for each method using 4 bonds for $\mathrm{RI}(1), 3$ bonds for HD and 5 bonds for KRD. Return errors are averaged at every time period over each $d$-day ahead forecast, where $d=1, \ldots, 100$. The time series plot in Figure 4a shows the 180 -day moving average for visibility.

## 5 Evaluation: dynamic hedging

Although the static hedging experiment in Section 4 may be informative, it only addresses the performance of various immunization methods under a one-shot instantaneous change in the yield curve. In practice, the fund manager will rebalance the portfolio over time, in which case the yield curve as well as the bond maturities change. In this section, we conduct a dynamic hedging experiment using simulated yield curves.

Table 2: $\ell^{1}$ norm of investment shares.

| Method: | RI(0) | RI(1) | RI(2) | HD | KRD |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Median |  |  |  |  |  |
| $J=2$ | 1 | 1 | 1 | 1 | 1 |
| $J=3$ | 1 | 1 | 1 | 1 | 1 |
| $J=4$ | 1 | 1 | 1 | 5.49 | 1 |
| $J=5$ | 1 | 1 | 1.09 | 15.55 | 1 |
| $95^{\text {th }}$ percentile |  |  |  |  |  |
| $J=2$ | 1 | 1 | 1 | 1 | 1 |
| $J=3$ | 1.02 | 1 | 1.27 | 1.27 | 1 |
| $J=4$ | 1 | 1 | 1.18 | 12.12 | 1 |
| $J=5$ | 1 | 1 | 1.91 | 30.07 | 1.01 |
| 99 ${ }^{\text {th }}$ percentile |  |  |  |  |  |
| $J=2$ | 1 | 1 | 1 | 1 | 1 |
| $J=3$ | 1.05 | 1 | 1.31 | 1.31 | 1 |
| $J=4$ | 1 | 1 | 1.17 | 13.69 | 1 |
| $J=5$ | 1 | 1 | 2.12 | 33.6 | 1.05 |

Note: This table shows the $\ell^{1}$ norm of the investment shares, $\|\theta\|_{1}$, for robust immunization with a value matching constraint $(\mathrm{RI}(0))$, robust immunization with a value- and duration matching constraint (RI(1)), robust immunization with a value-, duration- and convexity matching constraint (RI(2)), high-order duration matching (HD) and key rate duration matching (KRD).

### 5.1 Implementing dynamic hedging

Let $\left\{s_{n}\right\}_{n=0}^{N}$ be the portfolio rebalancing dates (with the normalization $s_{0}=0$ ) and assume that the coupon payment dates of the liability are contained in this set. For simplicity let $s_{n}=n \Delta$ with $\Delta>0$ so the dates are evenly spaced, although this is inessential. The liability pays $f_{n} \geq 0$ at $s_{n}>0$. The fund manager can use $J$ zero-coupon bonds with face value 1 and maturities $\left\{t_{j}\right\}_{j=1}^{J}$ to hedge the liability. We introduce the following notations:

$$
\begin{aligned}
x_{s}(t) & =\text { cumulative discount rate for term } t \text { at time } s, \\
P_{s} & =\text { present value of liability at time } s, \\
V_{s} & =\text { net asset value (NAV) of fund at time } s, \\
z_{s}=\left(z_{s j}\right) & =\text { immunizing portfolio at time } s, \\
C_{s} & =\text { cash position at time } s, \\
R_{s} & =\text { gross short rate at time } s .
\end{aligned}
$$

We now describe how to calculate these quantities recursively. At time $s$, the
present value of the liability (after coupon payment) is

$$
P_{s}:=\sum_{n: s_{n}>s} \mathrm{e}^{-x_{s}\left(s_{n}-s\right)} f_{n} .
$$

Note that at time $s$, the remaining term of the $n$-th payment is $s_{n}-s$ and we only retain future payments in the sum. Let $s^{-}=s-\Delta$ denote the previous rebalancing period. The NAV of the fund consists of the present value of the bond and cash positions carried over from the previous period minus the current liability payment, which is

$$
V_{s}:=R_{s^{-}} C_{s^{-}}+\sum_{j=1}^{J} z_{s^{-} j} \mathrm{e}^{-x_{s}\left(t_{j}-\Delta\right)}-f_{s} .
$$

Here, note that the cash position earns a (predetermined) gross return $R_{s^{-}}$, and the zero-coupon bonds have shorter maturities $t_{j}-\Delta$ because time has passed. The equity (assets minus liabilities) is therefore

$$
\begin{align*}
E_{s} & :=V_{s}-P_{s} \\
& =R_{s^{-}} C_{s^{-}}+\sum_{j=1}^{J} z_{s^{-} j} \mathrm{e}^{-(x+h)\left(t_{j}-\Delta\right)}-f_{s}-\sum_{n: s_{n}>s} \mathrm{e}^{-(x+h)\left(s_{n}-s\right)} f_{n} \\
& =R_{s^{-}} C_{s^{-}}-f_{s}+\sum_{j=1}^{J} z_{s^{-} j} \mathrm{e}^{-(x+h)\left(t_{j}-\Delta\right)}-\sum_{n: s_{n}-\Delta_{-s^{-}}>0} \mathrm{e}^{-(x+h)\left(s_{n}-\Delta-s^{-}\right)} f_{n}, \tag{5.1}
\end{align*}
$$

where $x=x_{s^{-}}$denotes the cumulative discount rate at $s^{-}$and $h=x_{s}-x_{s^{-}}$ denotes the perturbation in the cumulative discount rate. As an illustration, consider the robust immumization method introduced in Section 3. The fund manager's problem at time $s^{-}$is to maximize the worst case equity, where the equity is defined by $E_{s}$ in (5.1). Shifting $s^{-}$to $s$, the time $s$ objective function is then

$$
E_{s+\Delta}(x ; h):=R_{s} C_{s}-f_{s+\Delta}+\sum_{j=1}^{J} z_{s j} \mathrm{e}^{-(x+h)\left(t_{j}-\Delta\right)}-\sum_{n: s_{n}-\Delta-s>0} \mathrm{e}^{-(x+h)\left(s_{n}-\Delta-s\right)} f_{n},
$$

where $x=x_{s}$ is the current cumulative discount rate. Because $f_{s+\Delta}$ is predetermined and $C_{s}$ is determined by the budget constraint and hence independent of the perturbation $h$, the dynamic hedging problem reduces to the static hedging problem discussed in Section 3 except that all payments need to be treated as if their maturities are reduced by $\Delta$. This modification takes into account the pas-
sage of time and hence the reduction in bond maturities by the next rebalancing date. For example, if the time to rebalancing is one month, a 1-year zero coupon bond is treated as if it is an 11-month bond.

Given the current cumulative discount rate $x_{s}$, it is straightforward to apply various immunizing methods to bonds and liability with maturities reduced by $\Delta$. Suppose the new (time $s$ ) immunizing portfolio $z_{s}=\left(z_{s j}\right)$ is chosen. Then the cash position is the difference between the NAV and portfolio value, which is

$$
C_{s}=V_{s}-\sum_{j=1}^{J} z_{s j} \mathrm{e}^{-x_{s}\left(t_{j}\right)} .
$$

Note that although we reduce the maturities by $\Delta$ to form the portfolio, we use the actual maturities to evaluate the portfolio value and define the cash position. Initializing at $V_{0}=P_{0}$ ( $100 \%$ funding), we can implement dynamic hedging by repeating this procedure for $s=\Delta, 2 \Delta, \ldots$. The funding ratio at time $s$ after coupon payments is then

$$
\begin{equation*}
\frac{V_{s}}{P_{s}} . \tag{5.2}
\end{equation*}
$$

### 5.2 Experimental design

Yield curve model In Section 4, we used the parsimonious Svensson (1994) model fitted to the historical yield curve data to evaluate the performance of static hedging. Unlike static hedging, where we only consider changes to the yield curve over short horizons, in dynamic hedging the yield curve changes over long horizons have a large impact on portfolio performance. This feature makes it problematic to use historical data for performance evaluation. For instance, suppose that a particular portfolio selection method over-weights in long-term bonds. Because historical yields have been trending downwards during the 1985-2022 period, this method may appear to have a good performance. However, the opposite is true had the yields been trending upwards.

For this reason, in our dynamic hedging experiment, we only use simulated yield curves generated from a no-arbitrage term structure model. Specifically, we apply the Ang et al. (2008) 3-factor regime switching model. By simulating yields from this (stationary) regime-switching model, we can analyze the efficacy of each immunization method under a wide variety of different yield curves. A more detailed description of the model, as well as the data used to estimate the model is provided in Appendix D. ${ }^{11}$

[^8]We implement the dynamic hedging approach using the same liability and zero-coupon bonds from the static problem in Section 4.1. We use all 5 bonds for immunization for $\operatorname{RI}(0), \operatorname{RI}(1)$ and KRD . In contrast, we only use 3 bonds for $H D$ since the performance for $J>3$ is comparatively worse relative to the other methods (see Figure 3). Since we estimate the yield curve model of Ang et al. (2008) based on quarterly data, we assume that the immunizing portfolio is rebalanced every quarter. We analyze the performance over a period of 10 years and repeat the simulation 5,000 times.

Results The results are summarized in Figure 5. The left panel shows the distribution of funding ratios at the end of the 10-year period across all simulations. Overall, it is clear that $\operatorname{RI}(1)$ is the superior method, since it is centered around 1 and has the smallest variance. Also, the MSE is an order of magnitude smaller compared to KRD, which comes second best. RI ( 0 ) is third best in terms of MSE, and is centered around 1 , even though the variance is much higher than $\operatorname{RI}(1)$ or KRD. The worst performing method is HD, whose distribution is characterized by large outliers in the left and right tail and the MSE is twice as high relative to RI(0).

The right panel of Figure 5 sheds light on the maxmin property by showing the first percentile of the funding ratio for each method throughout the 10-year period across all simulations. We see that $\operatorname{RI}(1)$ strictly dominates the other methods in the maxmin sense as well, consistently maintaining a funding ratio above that of the competing methods. Even after 10-years of rebalancing, the RI(1) method has a funding ratio close to $99 \%$. While KRD is comparable to $\operatorname{RI}(0)$ and $\operatorname{RI}(1)$ in earlier periods, it performs better over longer horizons, achieving an end-of-period funding ratio equal to $98.5 \%$ in the $1 \%$ worst-case scenario.

## 6 Conclusion

This paper uses techniques from functional and numerical analysis to study the classical portfolio immunization problem. The goal is to construct a portfolio that protects a financial institution against interest rate risk. We use the concept of a Fréchet derivative to find a portfolio that hedges against general perturbations to the cumulative discount rate. Subsequently, we present a maxmin result that proves existence of an immunizing portfolio which maximizes the worst-case equity loss and we provide a solution algorithm. This maxmin portfolio, which we

[^9]

Figure 5: Distribution of funding ratio
The left panel shows the empirical density of funding ratios calculated at the end of the 10-year immunization period. The right panel shows the first percentile of the funding ratio throughout the 10-year immunization period, calculated across all 5,000 simulations.
refer to as robust immunization, contains duration and convexity matching as a special case. In our empirical applications, we show that a judicious choice of basis functions for the discount rate leads to a robust immunization method that outperforms existing approaches in the static and dynamic case.

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## A Space of cumulative discount rates

Lemma A.1. Let $\mathcal{X}=\left\{x \in C^{1}[0, T]: x(0)=0\right\}$ be the vector space of continuously differentiable functions on $[0, T]$ with $x(0)=0$. For $x \in \mathcal{X}$, define $\|x\|_{\mathcal{X}}=\sup _{t \in[0, T]}\left|x^{\prime}(t)\right|$. Then $\left(\mathcal{X},\|\cdot\|_{\mathcal{X}}\right)$ is a Banach space.

Proof. Write $\|\cdot\|=\|\cdot\|_{\mathcal{X}}$ to simplify notation. Let us first show that $\|\cdot\|$ is a norm on $\mathcal{X}$. Since $x \in \mathcal{X}$ is continuously differentiable, $x^{\prime}$ is continuous, so $\|x\|=\sup _{t \in[0, T]}\left|x^{\prime}(t)\right| \in[0, \infty)$. Clearly $0 \in \mathcal{X}$ and $\|0\|=0$. If $\|x\|=0$, then $x^{\prime}(t)=0$ for all $t \in[0, T]$. Then $x(t)=x(0)+\int_{0}^{t} x^{\prime}(u) \mathrm{d} u=0$ because $x(0)=0$, so $x=0$. For any $\alpha \in \mathbb{R}$ and $x \in \mathcal{X}$, we have

$$
\|\alpha x\|=\sup _{t \in[0, T]}\left|\alpha x^{\prime}(t)\right|=|\alpha| \sup _{t \in[0, T]}\left|x^{\prime}(t)\right|=|\alpha|\|x\| .
$$

For anly $x, y \in \mathcal{X}$, we have

$$
\|x+y\|=\sup _{t \in[0, T]}\left|x^{\prime}(t)+y^{\prime}(t)\right| \leq \sup _{t \in[0, T]}\left|x^{\prime}(t)\right|+\sup _{t \in[0, T]}\left|y^{\prime}(t)\right|=\|x\|+\|y\| .
$$

Therefore $\|\cdot\|$ is a norm. To show that $\mathcal{X}$ is complete, let $\left\{x_{n}\right\}_{n=1}^{\infty} \subset \mathcal{X}$ be a Cauchy sequence with respect to the norm $\|\cdot\|$. Then by the definition of $\|\cdot\|,\left\{x_{n}^{\prime}\right\}_{n=1}^{\infty}$ is Cauchy in $C[0, T]$, so there exists $f \in C[0, T]$ such that $\left\|x_{n}^{\prime}-f\right\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$, where $\|\cdot\|_{\infty}$ denotes the supremum norm in $C[0, T]$. Define $x(t)=\int_{0}^{t} f(u) \mathrm{d} u$. Then clearly $x$ is continuously differentiable and $x(0)=0$, so $x \in \mathcal{X}$. Furthermore,

$$
\left\|x_{n}-x\right\|=\sup _{t \in[0, T]}\left|x_{n}^{\prime}(t)-x^{\prime}(t)\right|=\sup _{t \in[0, T]}\left|x_{n}^{\prime}(t)-f(t)\right|=\left\|x_{n}^{\prime}-f\right\|_{\infty} \rightarrow 0
$$

so we have $x_{n} \rightarrow x$ in $\mathcal{X}$. Therefore $\left(\mathcal{X},\|\cdot\|_{\mathcal{X}}\right)$ is a Banach space.
Lemma A. 2 (Polynomial basis). Suppose Assumption 1 holds and $h_{i}$ is a polynomial of degree $i$ with $h_{i}(0)=0$. Then Assumption 3 holds.

Proof. Since $h_{i}$ is a polynomial of degree $i$ with $h_{i}(0)=0$, without loss of generality we may assume $h_{i}(t)=t^{i}$. Then $h_{i}^{\prime}(t)=i t^{i-1}$. By the Stone-Weierstrass theorem (Folland, 1999, p. 139), span $\left\{h_{i}^{\prime}\right\}_{i=1}^{\infty}$ is dense in $C[0, T]$. Since $\|x\|_{\mathcal{X}}=\left\|x^{\prime}\right\|_{\infty}$, it follows that span $\left\{h_{i}\right\}_{i=1}^{\infty}$ is dense in $\mathcal{X}$. By Assumption 1, we can choose $I$ distinct points $\left\{t_{n_{j}}\right\}_{j=1}^{I}$. Consider the $I \times I$ submatrix of $H$ defined by $\tilde{H}=\left(h_{i}\left(t_{n_{j}}\right)\right)=$ $\left(t_{n_{j}}^{i}\right)$. Dividing the $j$-th column by $t_{n_{j}}>0, \tilde{H}$ reduces to a Vandermonde matrix, which is invertible. Therefore $H$ has full row rank. The same argument applies to $G$.

Lemma A.3. Fix $x \in \mathcal{X}$ and define $T: \mathcal{X} \rightarrow \mathbb{R}$ by

$$
T h=\delta P(x ; h)=-\int_{0}^{T} \mathrm{e}^{-x(t)} h(t) \mathrm{d} F(t)
$$

Then $T$ is a bounded linear operator.

Proof. Clearly $T$ is a linear operator. If $h \in \mathcal{X}$, then $\|h\|_{\mathcal{X}}=\sup _{t \in[0, T]}\left|h^{\prime}(t)\right|<\infty$. Since $h(0)=0$, we obtain

$$
|h(t)|=\left|\int_{0}^{t} h^{\prime}(u) \mathrm{d} u\right| \leq \int_{0}^{t}\|h\|_{\mathcal{X}} \mathrm{d} u=t\|h\|_{\mathcal{X}} .
$$

Therefore

$$
|T h| \leq \int_{0}^{T} \mathrm{e}^{-x(t)}|h(t)| \mathrm{d} F(t) \leq\|h\|_{\mathcal{X}} \int_{0}^{T} t \mathrm{e}^{-x(t)} \mathrm{d} F(t),
$$

so $T$ is a bounded linear operator with $\|T\| \leq \int_{0}^{T} t \mathrm{e}^{-x(t)} \mathrm{d} F(t)$.

## B Proof of main results

Proof of Proposition 3.1. Let us first show that $\mathcal{W}$ in (3.5) is compact, convex, and contains 0 in the interior. Clearly $0 \in \mathcal{W}$. Since $w \mapsto G^{\prime} w$ is linear (hence continuous) and $G^{\prime} 0=0$ is an interior point of $[-1,1]^{N}, 0$ is an interior point of $\mathcal{W}$. Since $\mathcal{W}$ is defined by weak linear inequalities, it is closed and convex. Let us show compactness. By Assumption 3, $H$ has full row rank, and so does $G$. Take $n_{1}, \ldots, n_{I}$ such that the $I \times I$ matrix $\tilde{G}:=\left(g_{i, n_{j}}\right)$ is invertible. Define

$$
\tilde{\mathcal{W}}:=\left\{w \in \mathbb{R}^{I}: \tilde{G}^{\prime} w \in[-1,1]^{I}\right\}=\left(\tilde{G}^{\prime}\right)^{-1}[-1,1]^{I} .
$$

Since $\tilde{\mathcal{W}}$ is defined by a subset of inequalities that define $\mathcal{W}$, clearly we have $\mathcal{W} \subset \tilde{\mathcal{W}}$. Furthermore, $\tilde{\mathcal{W}}$ is compact because it is the image of the compact set $[-1,1]^{I}$ under the linear (hence continuous) map $\left(\tilde{G}^{\prime}\right)^{-1}: \mathbb{R}^{I} \rightarrow \mathbb{R}^{I}$. Therefore $\mathcal{W} \subset \tilde{\mathcal{W}}$ is compact.

Next, let us show that the minmax problem (3.6) has a solution $\left(z^{*}, w^{*}\right) \in$ $\mathcal{Z} \times \mathcal{W}$. Since $\mathcal{W}$ is nonempty and compact and $w \mapsto\langle w, A z-b\rangle$ is linear (hence continuous),

$$
\begin{equation*}
M(z):=\max _{w \in \mathcal{W}}\langle w, A z-b\rangle \tag{B.1}
\end{equation*}
$$

exists. The maximum theorem (Berge, 1963, p. 116) implies that $M$ is continuous. Furthermore, since $0 \in \mathcal{W}$, we have $M(z) \geq 0$ and hence $V_{I}(\mathcal{Z})=\inf _{z \in \mathcal{Z}} M(z) \geq 0$. Let $\|\cdot\|_{2}$ denote the $\ell^{2}$ (Euclidean) norm. Since 0 is an interior point of $\mathcal{W}$, there exists $\epsilon>0$ such that $w \in \mathcal{W}$ whenever $\|w\|_{2} \leq \epsilon$. If $A z \neq b$, setting $w=\epsilon \frac{A z-b}{\|A z-b\|_{2}}$, we obtain

$$
\begin{equation*}
M(z) \geq\left\langle\epsilon \frac{A z-b}{\|A z-b\|_{2}}, A z-b\right\rangle=\epsilon\|A z-b\|_{2} . \tag{B.2}
\end{equation*}
$$

Note that the lower bound (B.2) is valid even if $A z=b$.
To bound (B.2) from below, let us show that

$$
\begin{equation*}
\|A z-b\|_{2}=\left\|A_{+} z-b_{+}\right\|_{2} \tag{B.3}
\end{equation*}
$$

when $z \in \mathcal{Z}$. Using the definition (3.4), it suffices to show that $a_{0} z-1=0$ if $z \in \mathcal{Z}$. But since by Assumption 2 value matching holds, dividing (2.6) by $P(x)$ and using (3.2) for $i=0$ (hence $h_{0} \equiv 1$ ), we obtain

$$
1=\frac{1}{P(x)} \sum_{j=1}^{J} z_{j} P_{j}(x)=\sum_{j=1}^{J} a_{0 j} z_{j}=a_{0} z,
$$

which implies (B.3). Define $m:=\min _{\|z\|_{2}=1}\left\|A_{+} z\right\|_{2}$, which is achieved because $\|z\|_{2}=1$ is a nonempty compact set and $z \mapsto\left\|A_{+} z\right\|_{2}$ is continuous. Since by assumption $A_{+}$has full column rank, we have $A_{+} z=0$ only if $z=0$, so $m>0$. Therefore it follows from (B.2) and (B.3) that for any $z \in \mathcal{Z}$,

$$
\begin{equation*}
M(z) \geq \epsilon\|A z-b\|_{2}=\epsilon\left\|A_{+} z-b_{+}\right\|_{2} \geq \epsilon\left(m\|z\|_{2}-\left\|b_{+}\right\|_{2}\right) \rightarrow \infty \tag{B.4}
\end{equation*}
$$

as $\|z\|_{2} \rightarrow \infty$, so we may restrict the minimization of $M(z)$ to a compact subset of $\mathcal{Z}$. Since $M(z)$ is continuous, the minmax value $V_{I}(\mathcal{Z})$ is achieved.

Finally, let us show that $z \in \mathcal{Z}$ achieves $V_{I}(\mathcal{Z})=0$ if and only if $A_{+} z=b_{+}$. If $A_{+} z=b_{+}$, then $A z=b$ so clearly $M(z)=0$ and $V_{I}(\mathcal{Z})=0$. If $V_{I}(\mathcal{Z})=0$, then for any $z \in \mathcal{Z}$ with $M(z)=V_{I}(\mathcal{Z})=0$, (B.2) and (B.3) imply $\left\|A_{+} z-b_{+}\right\|_{2}=0$ and therefore $A_{+} z=b_{+}$.

Proof of Proposition 3.2. Suppose that $\operatorname{span}\left\{\tilde{h}_{i}\right\}_{i=1}^{I}=\operatorname{span}\left\{h_{i}\right\}_{i=1}^{I}$. Since $\left\{h_{i}\right\}_{i=1}^{I}$ span $\left\{\tilde{h}_{i}\right\}_{i=1}^{I}$, there exists an $I \times I$ matrix $C=\left(c_{i j}\right)$ such that $\tilde{h}_{i}=\sum_{j=1}^{I} c_{i j} h_{j}$. Since $\left\{h_{i}\right\}_{i=1}^{I}$ are linearly independent, $C$ is unique. Since $\left\{\tilde{h}_{i}\right\}_{i=1}^{I}$ also span $\left\{h_{i}\right\}_{i=1}^{I}$, C must be invertible. Then $\tilde{H}=C H, \tilde{A}=C A, \tilde{b}=C b, \tilde{G}=C G$, so setting $w=C^{\prime} \tilde{w}$, we obtain

$$
\tilde{M}(z):=\sup _{\tilde{w}: \tilde{G}^{\prime} \tilde{w} \in[-1,1]^{N}}\langle\tilde{w}, \tilde{A} z-\tilde{b}\rangle=\sup _{w: G^{\prime} w \in[-1,1]^{N}}\langle w, A z-b\rangle=: M(z) .
$$

Therefore the minimizers of $M$ and $\tilde{M}$ agree and the conclusion holds.
To prove Theorem 3.3, we recall Taylor's theorem with the integral form for the remainder term.

Lemma B. 1 (Taylor's theorem). Let $f \in C^{n+1}[0,1]$, so $f:[0,1] \rightarrow \mathbb{R}$ is $n+1$ times continuously differentiable. Then

$$
\begin{equation*}
f(1)=\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}+\int_{0}^{1} f^{(n+1)}(s) \frac{(1-s)^{n}}{n!} \mathrm{d} s . \tag{B.5}
\end{equation*}
$$

Proof. For $n=0$, (B.5) is obvious from the fundamental theorem of calculus:

$$
f(1)-f(0)=\int_{0}^{1} f^{\prime}(s) \mathrm{d} s
$$

Suppose (B.5) holds for some $n-1$ and consider $n$. Using integration by parts and the induction hypothesis, we obtain

$$
\begin{aligned}
\int_{0}^{1} f^{(n+1)}(s) \frac{(1-s)^{n}}{n!} \mathrm{d} s & =\left[f^{(n)}(s) \frac{(1-s)^{n}}{n!}\right]_{0}^{1}+\int_{0}^{1} f^{(n)}(s) \frac{(1-s)^{n-1}}{(n-1)!} \mathrm{d} s \\
& =-\frac{f^{(n)}(0)}{n!}+\left(f(1)-\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\right) \\
& =f(1)-\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!},
\end{aligned}
$$

so (B.5) holds for $n$.
Proof of Theorem 3.3. For any $x, h \in \mathbb{R}$, define $f:[0,1] \rightarrow \mathbb{R}$ by $f(s)=\mathrm{e}^{-x-s h}$. Applying Lemma B. 1 for $n=1$, we obtain

$$
\mathrm{e}^{-x-h}=\mathrm{e}^{-x}-\mathrm{e}^{-x} h+\int_{0}^{1}(1-s) \mathrm{e}^{-x-s h} h^{2} \mathrm{~d} s
$$

Setting $x=x(t)$ and $h=h(t)$ for $x, h \in \mathcal{X}$ and integrating both sides on $[0, T]$ with respect to $F$, we obtain

$$
\begin{aligned}
\int_{0}^{T} \mathrm{e}^{-x(t)-h(t)} \mathrm{d} F(t)=\int_{0}^{T} \mathrm{e}^{-x(t)} \mathrm{d} F(t) & -\int_{0}^{T} \mathrm{e}^{-x(t)} h(t) \mathrm{d} F(t) \\
& +\int_{0}^{T} \int_{0}^{1}(1-s) \mathrm{e}^{-x(t)-s h(t)} h(t)^{2} \mathrm{~d} s \mathrm{~d} F(t)
\end{aligned}
$$

Using the definition of $P$ and $P^{\prime}$, we obtain

$$
\begin{equation*}
P(x+h)=P(x)+P^{\prime}(x) h+\int_{0}^{T} \int_{0}^{1}(1-s) \mathrm{e}^{-x(t)-s h(t)} h(t)^{2} \mathrm{~d} s \mathrm{~d} F(t) . \tag{B.6}
\end{equation*}
$$

A similar equation holds for each $P_{j}$. Hence for any $z=\left(z_{j}\right) \in \mathbb{R}^{J}$ we have

$$
\begin{equation*}
P(x+h)-\sum_{j=1}^{J} z_{j} P_{j}(x+h)=E_{0}+E_{1}+E_{2}, \tag{B.7}
\end{equation*}
$$

where

$$
\begin{align*}
& E_{0}:=P(x)-\sum_{j=1}^{J} z_{j} P_{j}(x),  \tag{B.8a}\\
& E_{1}:=\left(P^{\prime}(x)-\sum_{j=1}^{J} z_{j} P_{j}^{\prime}(x)\right) h,  \tag{B.8b}\\
& E_{2}:=\int_{0}^{T} \int_{0}^{1}(1-s) \mathrm{e}^{-x(t)-s h(t)} h(t)^{2} \mathrm{~d} s \mathrm{~d}\left(F(t)-\sum_{j=1}^{J} z_{j} F_{j}(t)\right) . \tag{B.8c}
\end{align*}
$$

Since $\mathcal{Z}$ satisfies value matching by Assumption 2, we have $E_{0}=0$ by (B.8a). Inspection of Assumption 3, (3.8), and (3.5) reveals that any $h \in \mathcal{H}_{I}(\Delta)$ can be expressed as $h=\Delta \sum_{i=1}^{I} w_{i} h_{i}$ for some $w \in \mathcal{W}$. Using (B.8b), (3.2), and (3.3), we obtain

$$
\begin{equation*}
E_{1}=\left(P^{\prime}(x)-\sum_{j=1}^{J} z_{j} P_{j}^{\prime}(x)\right) h=\Delta P(x)\langle w, A z-b\rangle \tag{B.9}
\end{equation*}
$$

To bound $E_{2}$, note that the last integral in (B.6) is nonnegative because $1-s \geq$ 0 on $s \in[0,1]$ and $F$ is increasing. Furthermore, it can be bounded above by

$$
\int_{0}^{T} \int_{0}^{1}(1-s) \mathrm{e}^{-x(t)+\|h\|_{\infty}}\|h\|_{\infty}^{2} \mathrm{~d} s \mathrm{~d} F(t)=\frac{1}{2}\|h\|_{\infty}^{2} \mathrm{e}^{\|h\|_{\infty} P(x) .}
$$

Therefore $E_{2}$ in (B.8c) can be bounded as

$$
\begin{equation*}
-\frac{1}{2}\|h\|_{\infty}^{2} \mathrm{e}^{\|h\|_{\infty}} \sum_{z_{j} \geq 0} z_{j} P_{j}(x) \leq E_{2} \leq \frac{1}{2}\|h\|_{\infty}^{2} \mathrm{e}^{\|h\|_{\infty}}\left(P(x)-\sum_{z_{j}<0} z_{j} P_{j}(x)\right) \tag{B.10}
\end{equation*}
$$

Using (2.6) and (3.7), we obtain

$$
\begin{align*}
P(x)-\sum_{z_{j}<0} z_{j} P_{j}(x) & =\sum_{z_{j} \geq 0} z_{j} P_{j}(x)=\frac{1}{2}\left(P(x)+\sum_{j=1}^{J}\left|z_{j}\right| P_{j}(x)\right) \\
& =\frac{1}{2} P(x)\left(1+\sum_{j=1}^{J}\left|\theta_{j}\right|\right)=\frac{1}{2} P(x)\left(1+\|\theta\|_{1}\right) . \tag{B.11}
\end{align*}
$$

Noting that $\|h\|_{\infty} \leq \Delta T$ for $h \in \mathcal{H}_{I}(\Delta)$, it follows from (B.10) and (B.11) that

$$
\begin{equation*}
\left|E_{2}\right| \leq \frac{1}{4} \Delta^{2} T^{2} \mathrm{e}^{\Delta T} P(x)\left(1+\|\theta\|_{1}\right) \tag{B.12}
\end{equation*}
$$

Combining (B.7), $E_{0}=0$, (B.9), and (B.12), we obtain

$$
\begin{align*}
&\langle w, A z-b\rangle-\frac{1}{4} \Delta T^{2} \mathrm{e}^{\Delta T}\left(1+\|\theta\|_{1}\right) \\
& \leq \frac{1}{\Delta P(x)}\left[P(x+h)-\sum_{j=1}^{J} z_{j} P_{j}(x+h)\right] \\
& \quad \leq\langle w, A z-b\rangle+\frac{1}{4} \Delta T^{2} \mathrm{e}^{\Delta T}\left(1+\|\theta\|_{1}\right) \tag{B.13}
\end{align*}
$$

Since by (3.7) $\theta_{j}$ is proportional to $z_{j}$, there exists some constant $c(x)>0$ that depends only on $x$ such that $\|\theta\|_{1} \leq c(x)\|z\|_{2}$. Therefore maximizing (B.13) over $w \in \mathcal{W}$, it follows from the definition of $M(z)$ in (B.1) that

$$
\begin{align*}
M(z)-\frac{1}{4} \Delta T^{2} \mathrm{e}^{\Delta T}\left(1+c(x)\|z\|_{2}\right) & \\
\leq \frac{1}{\Delta P(x)} \sup _{h \in \mathcal{H}_{I}(\Delta)} & {\left[P(x+h)-\sum_{j=1}^{J} z_{j} P_{j}(x+h)\right] } \\
& \leq M(z)+\frac{1}{4} \Delta T^{2} \mathrm{e}^{\Delta T}\left(1+c(x)\|z\|_{2}\right) \tag{B.14}
\end{align*}
$$

Let $m, \epsilon>0$ be as in the proof of Proposition 3.1 and take $\bar{\Delta}>0$ such that $\epsilon m=\frac{1}{4} \bar{\Delta} T^{2} \mathrm{e}^{\bar{\Delta} T} c(x)$. Then if $0<\Delta<\bar{\Delta}$, by (B.4) both sides of (B.14) grow to infinity as $\|z\|_{2} \rightarrow \infty$. Therefore when we take the infimum of (B.14) as well as $M(z)$ with respect to $z \in \mathcal{Z}$, we may restrict it to some compact subset $\mathcal{Z}^{\prime} \subset \mathcal{Z}$. Therefore there exists a constant $c^{\prime}>0$ such that

$$
M(z)-c^{\prime} \Delta \leq \frac{1}{\Delta P(x)} \sup _{h \in \mathcal{H}_{I}(\Delta)}\left[P(x+h)-\sum_{j=1}^{J} z_{j} P_{j}(x+h)\right] \leq M(z)+c^{\prime} \Delta
$$

for all $z \in \mathcal{Z}^{\prime}$ and $\Delta \in(0, \bar{\Delta})$. Taking the infimum over $z \in \mathcal{Z}$ (which is achieved in $\mathcal{Z}^{\prime}$ ) and letting $\Delta \rightarrow 0$, by the definition of $V_{I}(\mathcal{Z})$ in (3.6), we obtain (3.9).

To show the error estimate (3.10), let $z^{*} \in \mathcal{Z}$ be a solution to the minmax problem (3.6). It follows from (B.13) that

$$
\frac{1}{\Delta P(x)}\left|P(x+h)-\sum_{j=1}^{J} z_{j}^{*} P_{j}(x+h)\right| \leq\left|\left\langle w, A z^{*}-b\right\rangle\right|+\frac{1}{4} \Delta T^{2} \mathrm{e}^{\Delta T}\left(1+\|\theta\|_{1}\right) .
$$

Taking the supremum over $w \in \mathcal{W}$ and noting that $\mathcal{W}$ is symmetric $(w \in \mathcal{W}$ implies $-w \in \mathcal{W}$ ), it follows from the definition of $V_{I}(\mathcal{Z})$ in (3.6) that (3.10) holds.

Proof of Proposition 3.4. For each $I$, let $M_{I}(z)=\sup _{w \in \mathcal{W}_{I}}\left\langle w, A_{I} z-b_{I}\right\rangle$, where $A_{I}, b_{I}$ denote the matrix $A$ and vector $b$ defined by (3.2) and (3.3) and $\mathcal{W}_{I}$ denotes the set $\mathcal{W}$ defined by (3.5). Suppose $I<I^{\prime}$. Letting $0_{N}$ denote the zero vector of $\mathbb{R}^{N}$, we have $\mathcal{W}_{I} \times\left\{0_{I^{\prime}-I}\right\} \subset \mathcal{W}_{I^{\prime}}$, so

$$
\begin{aligned}
M_{I}(z) & =\sup _{w \in \mathcal{W}_{I}}\left\langle w, A_{I} z-b_{I}\right\rangle=\sup _{w \in \mathcal{W}_{I} \times\left\{0_{I^{\prime}-I}\right\}}\left\langle w, A_{I^{\prime}} z-b_{I^{\prime}}\right\rangle \\
& \leq \sup _{w \in \mathcal{W}_{I^{\prime}}}\left\langle w, A_{I^{\prime}} z-b_{I^{\prime}}\right\rangle=M_{I^{\prime}}(z) .
\end{aligned}
$$

Taking the infimum over $z \in \mathcal{Z}$, we obtain $V_{I}(\mathcal{Z}) \leq V_{I^{\prime}}(\mathcal{Z})$. Similarly,

$$
V_{I}(\mathcal{Z})=\inf _{z \in \mathcal{Z}} M_{I}(z) \geq \inf _{z \in \mathcal{Z}^{\prime}} M_{I}(z)=V_{I}\left(\mathcal{Z}^{\prime}\right)
$$

Proof of Theorem 3.5. Because the proof is similar to that of Theorem 3.3, we only provide a sketch.

By assumption, $\mathcal{Z}_{1}$ in (3.12) is nonempty, and it is clearly closed. Hence by Proposition 3.1 the minmax value $V_{I}\left(\mathcal{Z}_{1}\right)$ defined by (3.6) is achieved by some $z^{*} \in$ $\mathcal{Z}_{1}$. Inspection of Assumption 3, (3.11), and (3.5) reveals that any $h \in \mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right)$ can be expressed as $h=\Delta_{1} v h_{1}+\Delta_{2} \sum_{i=1}^{I} w_{i} h_{i}$ for some $w \in \mathcal{W}$ and $v \in \mathbb{R}$ with $|v| \leq \min _{n} 1 /\left|h^{\prime}\left(t_{n}\right)\right|=: \bar{v} \in(0, \infty)$. Applying a similar argument to the derivation of (B.13), we obtain

$$
\begin{aligned}
& \frac{1}{P(x)}\left[P(x+h)-\sum_{j=1}^{J} z_{j} P_{j}(x+h)\right] \\
&=\Delta_{1} v(A z-b)_{1}+\Delta_{2}\langle w, A z-b\rangle+O\left(\Delta_{1}^{2}+\Delta_{2}^{2}\right),
\end{aligned}
$$

where $(A z-b)_{1}$ denotes the first entry of the vector $A z-b$. Maximizing both sides over $h \in \mathcal{H}_{I}\left(\Delta, \Delta_{1}\right)$, we obtain

$$
\begin{aligned}
& \sup _{h \in \mathcal{H}_{I}\left(\Delta, \Delta_{1}\right)} \frac{1}{P(x)}\left[P(x+h)-\sum_{j=1}^{J} z_{j} P_{j}(x+h)\right] \\
&=\Delta_{1} \bar{v}\left|(A z-b)_{1}\right|+\Delta_{2} M(z)+O\left(\Delta_{1}^{2}+\Delta_{2}^{2}\right)
\end{aligned}
$$

where $M(z)$ is defined by (B.1). Dividing both sides by $\Delta_{2}>0$ and letting
$\Delta_{2} \rightarrow 0, \Delta_{1} / \Delta_{2} \rightarrow \infty$, and $\Delta_{1}^{2} / \Delta_{2} \rightarrow 0$, the objective function remains finite only if $(A z-b)_{1}=0$, which is equivalent to $z \in \mathcal{Z}_{1}$. Under this condition, we have

$$
\frac{1}{\Delta} \sup _{h \in \mathcal{H}_{I}\left(\Delta_{1}, \Delta_{2}\right)} \frac{1}{P(x)}\left[P(x+h)-\sum_{j=1}^{J} z_{j} P_{j}(x+h)\right]=M(z)+O\left(\Delta_{2}+\Delta_{1}^{2} / \Delta_{2}\right)
$$

Minimizing over $z \in \mathcal{Z}_{1}$ and letting $\Delta_{2} \rightarrow 0$, we obtain (3.13). The proof of (3.14) is similar.

Proof of Proposition 3.7. Suppose that the liability has maturity $s$ with face value 1. Then the value of the liability is

$$
P(x)=\int_{0}^{T} \mathrm{e}^{-x(t)} \mathrm{d} F(t)=\mathrm{e}^{-x(s)}
$$

Let $z^{*}=A_{+}^{-1} b_{+}$be the immunizing portfolio and assume $z^{*} \geq 0$. Take any perturbation $h \in \operatorname{span}\left\{h_{i}\right\}_{i=1}^{I}$ and write $h=\sum_{i=1}^{I} w_{i} h_{i}$. Then the funding ratio is

$$
\phi(w):=\frac{\sum_{j=1}^{J} z_{j}^{*} P_{j}(x+h)}{P(x+h)}=\sum_{j=1}^{J} z_{j}^{*} \int_{0}^{T} \mathrm{e}^{-x(t)+x(s)-h(t)+h(s)} \mathrm{d} F_{j}(t) .
$$

Since $z^{*} \geq 0$ and the exponential function is convex, $\phi(w)$ is convex in $w \in \mathbb{R}^{I}$.
Let us show that $\nabla \phi(0)=0$. To this end we compute

$$
\begin{align*}
\frac{\partial \phi}{\partial w_{i}}(0) & =\sum_{j=1}^{J} z_{j}^{*} \int_{0}^{T} \mathrm{e}^{-x(t)+x(s)}\left(-h_{i}(t)+h_{i}(s)\right) \mathrm{d} F_{j}(t) \\
& =\mathrm{e}^{x(s)} \sum_{j=1}^{J} z_{j}^{*}\left(-\int_{0}^{T} \mathrm{e}^{-x(t)} h_{i}(t) \mathrm{d} F_{j}(t)+h_{i}(s) \int_{0}^{T} \mathrm{e}^{-x(t)} \mathrm{d} F_{j}(t)\right) \\
& =\mathrm{e}^{x(s)}\left(-P(x) \sum_{j=1}^{J} a_{i j} z_{j}^{*}+h_{i}(s) \sum_{j=1}^{J} z_{j}^{*} P_{j}(x)\right) \tag{B.15}
\end{align*}
$$

where the last line uses (3.2) and (2.4) for each bond $j$. Using value matching (2.6) and the fact that the liability is a zero-coupon bond, we obtain

$$
\begin{equation*}
h_{i}(s) \sum_{j=1}^{J} z_{j}^{*} P_{j}(x)=h_{i}(s) P(x)=\mathrm{e}^{-x(s)} h_{i}(s)=\int_{0}^{T} \mathrm{e}^{-x(t)} h_{i}(t) \mathrm{d} F(t)=P(x) b_{i}, \tag{B.16}
\end{equation*}
$$

where the last equality uses (3.3). Combining (B.15) and (B.16), we obtain

$$
\begin{equation*}
\nabla \phi(0)=b-A z^{*}=0 . \tag{B.17}
\end{equation*}
$$

Since $\phi$ is convex, it follows that $\phi(w) \geq \phi(0)=1$ for all $w$, which implies (3.17).

Proof of Lemma 3.8. For $i=1$, since $T_{0}(x)=1$, we have $g_{1}(t)=1$ and hence (3.19) implies (3.20a). For $i=2$, since $T_{1}(x)=x$, we have $g_{2}(t)=\min \{2 t / T-1,1\}$. Integrating this expression gives (3.20b). Suppose $i \geq 3$. Letting $x=\cos \theta$, we can evaluate the integral of Chebyshev polynomials as

$$
\begin{aligned}
\int_{-1}^{x} T_{n}(x) \mathrm{d} x & =\int_{\pi}^{\theta} \cos n \theta(-\sin \theta) \mathrm{d} \theta \\
& =\frac{1}{2} \int_{\pi}^{\theta}(\sin (n-1) \theta-\sin (n+1) \theta) \mathrm{d} \theta \\
& =\frac{1}{2}\left[\frac{\cos (n+1) \theta}{n+1}-\frac{\cos (n-1) \theta}{n-1}\right]_{\pi}^{\theta} \\
& =\frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1}-\frac{T_{n-1}(x)}{n-1}-\frac{2(-1)^{n}}{(n+1)(n-1)}\right) .
\end{aligned}
$$

Therefore for $i \geq 3$ we have

$$
h_{i}(t)=\frac{1}{4} T\left(\frac{T_{i}(2 t / T-1)}{i}-\frac{T_{i-2}(2 t / T-1)}{i-2}+\frac{2(-1)^{i}}{i(i-2)}\right),
$$

which is (3.20c)

## C Generic full column rank of $A_{+}$

This appendix shows that the matrix $A_{+}$in (3.4) generically has full column rank, which makes Proposition 3.1 applicable.

Proposition C.1. Let $I \geq J-1,\left\{h_{i}\right\}_{i=1}^{I}$ be the basis functions, and set $h_{0} \equiv 1$. Suppose that there exist $\left\{m_{i}\right\}_{i=1}^{J} \subset\{0,1, \ldots, I\}$ with $m_{1}=0$ and $\left\{\tau_{j}\right\}_{j=1}^{J} \subset(0, T]$ such that (i) at date $\tau_{j}$, bond $j$ makes a lump-sum payout $f_{j}:=F_{j}\left(\tau_{j}\right)-F_{j}\left(\tau_{j}-\right)>0$, and (ii) the $J \times J$ matrix $\tilde{H}=\left(h_{m_{i}}\left(\tau_{j}\right)\right)$ is invertible. Then there exists a closed set $S \subset \mathbb{R}^{J}$ with Lebesgue measure 0 such that the matrix $A_{+}$in (3.4) has full column rank whenever $\left(f_{1}, \ldots, f_{J}\right) \notin S$.

If in addition all bonds are zero-coupon bonds, then $A_{+}$has full column rank.

We need the following lemma to prove Proposition C.1.
Lemma C.2. Let $A, B$ be $N \times N$ matrices and define $f: \mathbb{R}^{N} \rightarrow \mathbb{R}$ by $f(x)=$ $\operatorname{det}(A \operatorname{diag}(x)+B)$, where $\operatorname{diag}(x)$ denotes the diagonal matrix with diagonal entries $x_{1}, \ldots, x_{N}$. If $\operatorname{det} A \neq 0$, then for any $c \in \mathbb{R}$ the set

$$
f^{-1}(c):=\left\{x \in \mathbb{R}^{N}: f(x)=c\right\}
$$

is closed and has Lebesgue measure 0.
Proof. Since

$$
\begin{aligned}
\operatorname{det}(A \operatorname{diag}(x)+B) & =\operatorname{det}\left(A\left(\operatorname{diag}(x)+A^{-1} B\right)\right) \\
& =\operatorname{det}(A) \times \operatorname{det}\left(\operatorname{diag}(x)+A^{-1} B\right)
\end{aligned}
$$

without loss of generality we may assume that $A$ is the identity matrix. Let $B=\left(b_{m n}\right)$. That $f^{-1}(c)$ is closed is obvious because $f$ is continuous.

Let us show by induction on the dimension $N$ that $f^{-1}(c)$ is a null set. If $N=1$, then $f(x)=x_{1}+b_{11}$, so $f^{-1}(c)=\left\{c-b_{11}\right\}$ is a singleton, which is a null set. Suppose the claim holds when $N=n-1$ and consider $n$. Let $B_{n}$ be the $n \times n$ matrix obtained from the first $n$ rows and columns of $B$, and let

$$
f_{n}\left(x_{1}, \ldots, x_{n}\right)=\operatorname{det}\left(\operatorname{diag}\left(x_{1}, \ldots, x_{n}\right)+B_{n}\right)
$$

Clearly $f_{n}$ is affine in each variable $x_{1}, \ldots, x_{n}$. Using the Laplace expansion along the $n$-th column, it follows that

$$
f_{n}\left(x_{1}, \ldots, x_{n}\right)=\left(x_{n}+b_{n n}\right) f_{n-1}\left(x_{1}, \ldots, x_{n-1}\right)+g_{n-1}\left(x_{1}, \ldots, x_{n-1}\right)
$$

for some function $g_{n-1}$ that is affine in each variable $x_{1}, \ldots, x_{n-1}$.
Define the sets $f_{n-1}^{-1}(0) \subset \mathbb{R}^{n-1}$ and $G \subset \mathbb{R}^{n}$ by

$$
\begin{aligned}
f_{n-1}^{-1}(0) & :=\left\{\left(x_{1}, \ldots, x_{n-1}\right): f_{n-1}\left(x_{1}, \ldots, x_{n-1}\right)=0\right\} \\
G & :=\left\{\left(x_{1}, \ldots, x_{n}\right):\left(x_{1}, \ldots, x_{n-1}\right) \notin f_{n-1}^{-1}(0), x_{n}=\left(c-g_{n-1}\right) / f_{n-1}-b_{n n}\right\} .
\end{aligned}
$$

Then $f_{n}^{-1}(c) \subset\left(f_{n-1}^{-1}(0) \times \mathbb{R}\right) \cup G$. By the induction hypothesis, $f_{n-1}^{-1}(0)$ has measure 0 in $\mathbb{R}^{n-1}$. Since $G$ is the graph of a Borel measurable function, by Fubini's theorem it has measure 0 . Therefore $f_{n}^{-1}(c)$ is a null set.

Proof of Proposition C.1. Define $\mathbf{h}:[0, T] \rightarrow \mathbb{R}^{I}$ by $\mathbf{h}(t)=\left(h_{0}(t), h_{1}(t), \ldots, h_{I}(t)\right)^{\prime}$.

Let the $j$-th column vector of $A_{+}$be $\mathbf{a}_{j}=\left(a_{0 j}, \ldots, a_{I j}\right)^{\prime}$. By assumption, bond $j$ pays $f_{j}>0$ at $\tau_{j} \in(0, T]$, so it follows from (3.2) that

$$
\begin{equation*}
\mathbf{a}_{j}=\frac{1}{P(x)} \int_{[0, T] \backslash\left\{\tau_{j}\right\}} \mathrm{e}^{-x(t)} \mathbf{h}(t) \mathrm{d} F_{j}(t)+\frac{1}{P(x)} \mathrm{e}^{-x\left(\tau_{j}\right)} f_{j} \mathbf{h}\left(\tau_{j}\right)=: \mathbf{p}_{j} f_{j}+\mathbf{q}_{j} \tag{C.1}
\end{equation*}
$$

Collecting (C.1) into a matrix, we can write $A_{+}=P \operatorname{diag}(f)+Q$, where $P, Q$ are matrices with $j$-th column vectors $\mathbf{p}_{j}, \mathbf{q}_{j}$ and $f=\left(f_{1}, \ldots, f_{J}\right)$. To show that $A_{+}$ generically has full column rank, let $\tilde{A}_{+}$be the $J \times J$ matrix obtained by taking its $m_{i}$-th row for $i=1, \ldots, J$. Define $\tilde{P}, \tilde{Q}$ similarly. Then $\tilde{A}_{+}=\tilde{P} \operatorname{diag}(f)+\tilde{Q}$. Since $\mathbf{p}_{j}=\mathrm{e}^{-x\left(\tau_{j}\right)} \mathbf{h}\left(\tau_{j}\right) / P(x)$, we obtain

$$
\operatorname{det} \tilde{P}=P(x)^{-J}\left(\prod_{j=1}^{J} \mathrm{e}^{-x\left(\tau_{j}\right)}\right) \operatorname{det} \tilde{H} \neq 0 .
$$

Therefore by Lemma C.2, $\tilde{A}_{+}$is generically invertible, so $A_{+}$has generically full column rank.

If in addition all bonds are zero-coupon bonds, then (C.1) reduces to $\mathbf{a}_{j}=$ $\mathrm{e}^{-x\left(\tau_{j}\right)} f_{j} \mathbf{h}\left(\tau_{j}\right) / P(x)$, where $\tau_{j}$ is the maturity. Then $A_{+}=P \operatorname{diag}(f)$, which has full column rank because $\operatorname{det} \tilde{P} \neq 0$ and $f_{j}>0$ for all $j$.

The fact that the set of $\left(f_{1}, \ldots, f_{J}\right)$ for which $A_{+}$has rank deficiency is contained in a closed set with Lebesgue measure 0 implies that the set of rank deficiency is nowhere dense (has empty interior). In this sense the rank deficiency of $A_{+}$is "rare". The following example shows that the zero-coupon bond assumption in Proposition C. 1 is essential.

Example 1. Suppose $I=J-1=1$ and the basis function is $h_{1}(t)=t$. Bond 1 is a zero-coupon bond with face value $f_{1}>0$ and maturity $t_{1}$. Bond 2 pays $f_{n}>0$ at time $t_{n}$, where $n=2,3$. To simplify notation, write $x\left(t_{1}\right)=x_{1}$ etc. The determinant of the matrix $A_{+}$is

$$
\begin{aligned}
\operatorname{det} A_{+} & =P(x)^{-2} \operatorname{det}\left[\begin{array}{cc}
f_{1} \mathrm{e}^{-x_{1}} & f_{2} \mathrm{e}^{-x_{2}}+f_{3} \mathrm{e}^{-x_{3}} \\
f_{1} \mathrm{e}^{-x_{1}} t_{1} & f_{2} \mathrm{e}^{-x_{2}} t_{2}+f_{3} \mathrm{e}^{-x_{3}} t_{3}
\end{array}\right] \\
& =P(x)^{-2} f_{1} \mathrm{e}^{-x_{1}}\left(f_{2} \mathrm{e}^{-x_{2}}\left(t_{2}-t_{1}\right)+f_{3} \mathrm{e}^{-x_{3}}\left(t_{3}-t_{1}\right)\right)
\end{aligned}
$$

Therefore for any $t_{2}<t_{1}<t_{3}$ and $f_{3}>0$, we have $\operatorname{det} A_{+}=0$ if and only if

$$
\begin{equation*}
\left(f_{1}, f_{2}\right) \in\left\{\left(f_{1}, f_{2}\right) \in \mathbb{R}_{++}^{2}: f_{2}=f_{3} \mathrm{e}^{x_{2}-x_{3}} \frac{t_{3}-t_{1}}{t_{1}-t_{2}}\right\} . \tag{C.2}
\end{equation*}
$$

The closure of the rank deficiency set (C.2) is a ray in $\mathbb{R}^{2}$ and has measure 0 .

## D No-arbitrage term structure model

The no-arbitrage term structure model of Ang et al. (2008) features multiple factors, regime switching, and closed-form solutions for bond prices, which is convenient to simulate yield curves. This appendix summarizes their model and presents parameter estimates based on our yield curve data.

## D. 1 Model and bond price formula

The equation numbers follow that of Ang et al. (2008). The model has three factors denoted by $X_{t}=\left(q_{t}, f_{t}, \pi_{t}\right)^{\prime}$. The dynamics of factors follows the regimedependent VAR process

$$
\begin{equation*}
X_{t+1}=\mu\left(s_{t+1}\right)+\Phi X_{t}+\Sigma\left(s_{t+1}\right) \varepsilon_{t+1} \tag{2}
\end{equation*}
$$

where

$$
\mu\left(s_{t}\right)=\left[\begin{array}{c}
\mu_{q}  \tag{3}\\
\mu_{f}\left(s_{t}\right) \\
\mu_{\pi}\left(s_{t}\right)
\end{array}\right], \quad \Phi=\left[\begin{array}{ccc}
\Phi_{q q} & 0 & 0 \\
\Phi_{f q} & \Phi_{f f} & 0 \\
\Phi_{\pi q} & \Phi_{\pi f} & \Phi_{\pi \pi}
\end{array}\right], \quad \Sigma\left(s_{t}\right)=\left[\begin{array}{ccc}
\sigma_{q} & 0 & 0 \\
0 & \sigma_{f}\left(s_{t}\right) & 0 \\
0 & 0 & \sigma_{\pi}\left(s_{t}\right)
\end{array}\right],
$$

and $\varepsilon$ is IID $N\left(0, I_{3}\right)$. The regime $s_{t}$ is a finite-state Markov chain taking values denoted by $k=1, \ldots, K$ with transition probability matrix $\Pi=\left(p_{k k^{\prime}}\right)$. The real short rate is given by

$$
\begin{equation*}
\widehat{r}_{t}=\delta_{0}+\delta_{1}^{\prime} X_{t} \tag{4}
\end{equation*}
$$

The regime-dependent price of risk is denoted by $\lambda\left(s_{t}\right)=\left(\lambda_{f}\left(s_{t}\right), \lambda_{\pi}\left(s_{t}\right)\right)^{\prime}$. Furthermore, define

$$
\begin{equation*}
\gamma_{t}=\gamma_{0}+\gamma_{1} q_{t}=\gamma_{0}+\gamma_{1} e_{1}^{\prime} X_{t} \tag{6}
\end{equation*}
$$

where $e_{n}$ denotes the $n$-th unit vector.
With this notation, the price of zero-coupon bonds can be obtained in closed form (Ang et al., 2008, Proposition B). For each maturity $n$, the nominal zerocoupon bond price in regime $i$ and factor $X$ is given by

$$
\begin{equation*}
P_{n}(i, X)=\exp \left(A_{n}(i)+B_{n} X\right) \tag{B1}
\end{equation*}
$$

where the scalar $A_{n}(i)$ and the $M \times 1$ vector $B_{n}$ can be computed as follows.
Let $M=3$ be the number of factors and $M_{1}=2$ be the number of non- $q$ factors. Partition $B_{n}$ as $B_{n}=\left[B_{n q} ; B_{n x}\right]$, where $B_{n q}$ is a scalar and $B_{n x}$ is $2 \times 1$. Similarly, let $\Sigma_{x}(i)$ be the lower $2 \times 2$ block of $\Sigma(i)$.

First, define $A_{0}(i)=0$ and $B_{0}=0$. Then define $\left\{\left(A_{n}, B_{n}\right)\right\}_{n=1}^{\infty}$ recursively by

$$
\begin{align*}
A_{n+1}(i) & =-\delta_{0}-B_{n q} \sigma_{q} \gamma_{0}+\log \sum_{j} p_{i j} \exp \left(A_{n}(j)+\left(B_{n}-e_{M}\right)^{\prime} \mu(j)\right. \\
& \left.-\left(B_{n x}-e_{M_{1}}\right)^{\prime} \Sigma_{x}(j) \lambda(j)+\frac{1}{2}\left(B_{n}-e_{M}\right)^{\prime} \Sigma(j) \Sigma(j)^{\prime}\left(B_{n}-e_{M}\right)\right),  \tag{B2.a}\\
B_{n+1} & =-\delta_{1}+\Phi^{\prime}\left(B_{n}-e_{M}\right)-B_{n q} \sigma_{q} \gamma_{1} e_{1} . \tag{B2.b}
\end{align*}
$$

## D. 2 Data

We use end of the quarter yield data from the U.S. Department of the Treasury for the period of 1990:Q1 to 2021:Q4; a total of 128 quarterly observations. ${ }^{12}$ The inflation data for the same period are obtained from the Bureau of Labor Statistics, from the CPI for All Urban Consumers series (seasonally adjusted).

In the model of Ang et al. (2008), there is a distinction between yields observed with and without error. Specifically, the yields observed with error take the form

$$
y_{n}(i, X)=-\frac{1}{n} \log P_{n}(i, X)+u
$$

where $u \sim N(0, V)$ is the error term with a diagonal covariance matrix $V$. To estimate the model, we use the 3 -month and 30 -year yields as observed without measurement error and we use the 10- and 50 -year yield as observed with error. However, the Treasury does not provide the 50 -year yield in the available data, which only extends to a maximum maturity of 30 years. Thus, we use the $30-$ year yield as a proxy for the 50 -year yield observed with measurement error. We find that including the 50 -year yield in the estimation is crucial to generate yield curves that remain relatively "flat" over long horizons, preventing the possibility of counterfactual steep declines at the long end of the yield curve.

## D. 3 Parameter estimates

We consider the benchmark model IV $^{\mathrm{C}}$ of Ang et al. (2008, §I.B.4). This model has four regimes. There are two state variables denoted by $s^{f}, s^{\pi}$, which both take

[^10]values in $\{1,2\}$. The combined state $s$ thus takes four values
\[

$$
\begin{aligned}
& s=1:=\left(s^{f}=1, s^{\pi}=1\right), \\
& s=2:=\left(s^{f}=1, s^{\pi}=2\right), \\
& s=3:=\left(s^{f}=2, s^{\pi}=1\right), \\
& s=4:=\left(s^{f}=2, s^{\pi}=2\right) .
\end{aligned}
$$
\]

We also impose the following restrictions consistent with Ang et al. (2008):

$$
\begin{align*}
\delta_{0} & =0.0069 \quad \text { (mean of nominal short rate) } \\
\delta_{1} & =\left(1,1, \delta_{\pi}\right)^{\prime} \\
\Phi_{f q} & =0  \tag{D.2}\\
\mu_{q} & =0 \\
\gamma_{0} & =0 \\
\lambda_{\pi}\left(s_{t}\right) & =0 .
\end{align*}
$$

We estimate the model using maximum likelihood, but numerical optimization is challenging since there are 36 parameters to estimate after imposing the restrictions in (D.2). In order to find the global maximum, we use the Global Optimization Toolbox from MATLAB to optimize the likelihood function for 20,000 different starting values. We then use the parameters that generate the highest likelihood value as starting values to optimize the likelihood function using the fminunc algorithm from MATLAB. Table 3 below summarizes the resulting parameter estimates.

Table 3: Parameter estimates

| Real short rate |  | $\delta_{1}$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $\delta_{0}$ | $q$ | $f$ | $\pi$ |
|  | 0.007 | 1.000 | 1.000 | -0.906 |
| Companion Form $\Phi$ |  | $q$ | $f$ | $\pi$ |
|  | $q$ | 0.972 | 0.000 | 0.000 |
|  | $f$ | 0.000 | 0.926 | 0.000 |
|  | $\pi$ | -0.733 | 0.991 | 0.840 |
| Moments of $X_{t}$ |  |  |  |  |
|  |  |  | Regime 1 | Regime 2 |
|  |  | $\mu_{q} \times 100$ | 0.000 | 0.000 |
|  |  | $\mu_{f}\left(s_{t}^{f}\right) \times 100$ | -0.025 | -0.009 |
|  |  | $\mu_{\pi}\left(s_{t}^{\pi}\right) \times 100$ | 0.004 | 0.790 |
|  |  | $\sigma_{q} \times 100$ | 0.109 | 0.109 |
|  |  | $\sigma_{f}\left(s_{t}^{f}\right) \times 100$ | 0.078 | 0.003 |
|  |  | $\sigma_{\pi}\left(s_{t}^{\pi}\right) \times 100$ | 0.001 | 0.948 |
| Risk parameters |  |  | $\lambda_{f}\left(s_{t}^{\pi}\right)$ |  |
|  |  | $\gamma_{1}$ | Regime 1 | Regime 2 |
|  |  | -25.042 | -1.187 | -0.192 |
| Transition Probability $\Pi$ |  |  |  |  |
|  | $s_{t+1}=1$ | $s_{t+1}=2$ | $s_{t+1}=3$ | $s_{t+1}=4$ |
| $s_{t}=1$ | 0.973 | 0.026 | 0.001 | 0.001 |
| $s_{t}=2$ | 0.003 | 0.920 | 0.003 | 0.074 |
| $s_{t}=3$ | 0.000 | 0.922 | 0.051 | 0.026 |
| $s_{t}=4$ | 0.107 | 0.310 | 0.515 | 0.068 |

Note: This table shows parameter estimates from the regime switching model of Ang et al. (2008). The estimated standard deviation of the unobserved yield errors for the 10- and 50 -year maturity yields is $0.155 \mathrm{e}-02$ and $0.344 \mathrm{e}-02$ respectively.

## E Miscellaneous results

## E. 1 Bias in the estimated yield curve

In our empirical application in Section 4, we assume that the forward rate is constant beyond the 30-year maturity, $f(t)=f(30)$ for all $t \geq 30$. As a result, the inferred date $s$ yield curve with term $t \geq 30$ satisfies $^{13}$

$$
\begin{aligned}
\widehat{y}_{s}(t) & :=\frac{1}{t} \int_{0}^{t} f_{s}(u) \mathrm{d} u=\frac{1}{t} \int_{0}^{30} f_{s}(u) \mathrm{d} u+\frac{1}{t} \int_{30}^{t} f_{s}(30) \mathrm{d} u \\
& =\frac{1}{t} \int_{0}^{30} f_{s}(u) \mathrm{d} u+f_{s}(30)-\frac{30}{t} f_{s}(30) \\
& =f_{s}(30)+O\left(\frac{1}{t}\right)
\end{aligned}
$$

Taking unconditional expectations and comparing to the true (unobserved) yield, we obtain

$$
\begin{align*}
\mathbb{E}\left[\widehat{y}_{s}(t)-y_{s}(t)\right] & =\mathbb{E}\left[f_{s}(30)-y_{s}(t)\right]+O\left(\frac{1}{t}\right) \\
& =\mathbb{E}\left[f_{s}(30)-f_{s}(t)\right]+\mathbb{E}\left[f_{s}(t)-y_{s}(t)\right]+O\left(\frac{1}{t}\right) . \tag{E.1}
\end{align*}
$$

Under integrability conditions on $y_{s}(t)$ and a mild stationarity assumption on bond returns, Alvarez and Jermann (2005, Proposition 5) show that

$$
\begin{equation*}
\mathbb{E}\left[\lim _{t \rightarrow \infty} f_{s}(t)\right]=\mathbb{E}\left[\lim _{t \rightarrow \infty} y_{s}(t)\right] \tag{E.2}
\end{equation*}
$$

Using the dominated convergence theorem and (E.2) in (E.1), we get

$$
\mathbb{E}\left[\widehat{y}_{s}(t)\right]=\mathbb{E}\left[y_{s}(t)\right]+\mathbb{E}\left[f_{s}(30)-f_{s}(t)\right]+o(1)
$$

Hence, on average we estimate the correct yield plus a bias term that reflects the average gap between the 30-year forward rate and long forward rate.

## E. 2 Approximating forward rate changes

In this appendix we evaluate the goodness-of-fit of approximating forward rate changes by basis functions. Let $I$ be the number of basis functions to include, $d$

[^11]be the number of days ahead, and $\left\{t_{n}\right\}_{n=1}^{N}$ be the set of terms (in years) to evaluate forward rates, where we set $t_{n}=n / 12$ and $N=360$ so that it corresponds to a 30 -year horizon at monthly interval. We use the following procedure.
(i) For each day $s$ and term $t_{n}$, calculate the $d$-day ahead change in the forward rate $f_{s+d}\left(t_{n}\right)-f_{s}\left(t_{n}\right)$ by evaluating (4.1).
(ii) Estimate
\[

$$
\begin{equation*}
f_{s+d}\left(t_{n}\right)-f_{s}\left(t_{n}\right)=\sum_{i=1}^{I} \gamma_{i s} g_{i}\left(t_{n}\right)+\epsilon_{s}\left(t_{n}\right) \tag{E.3}
\end{equation*}
$$

\]

by ordinary least squares (OLS), where $g_{i}$ is the basis function for the forward rate in (3.18). Let $\widehat{\gamma}_{i s}$ be the OLS estimator.
(iii) Calculate the goodness-of-fit measure

$$
\begin{equation*}
R^{2}:=\frac{\sum_{s=1}^{S} \sum_{n=1}^{N}\left(\sum_{i=1}^{I} \widehat{\gamma}_{i s} g_{i}\left(t_{n}\right)\right)^{2}}{\sum_{s=1}^{S} \sum_{n=1}^{N}\left(f_{s+d}\left(t_{n}\right)-f_{s}\left(t_{n}\right)\right)^{2}} \tag{E.4}
\end{equation*}
$$

The goodness-of-fit measure (E.4) is similar to the conventional $R^{2}$ in OLS, except that we use " 0 " as the benchmark instead of the sample mean because $g_{1} \equiv 1$ is already constant. The following proposition shows that $R^{2}$ in (E.4) can be computed efficiently.

Proposition E. 1 (Efficient calculation of $R^{2}$ ). Define the $S \times N$ matrix $C=\left(c_{s n}\right)$ by $c_{s n}=f_{s+d}\left(t_{n}\right)-f_{s}\left(t_{n}\right)$ and the $I \times N$ matrix $G=\left(g_{i}\left(t_{n}\right)\right)$. Then

$$
\begin{equation*}
R^{2}=\frac{\operatorname{tr}\left(G^{\prime}\left(G G^{\prime}\right)^{-1} G C^{\prime} C\right)}{\operatorname{tr}\left(C^{\prime} C\right)} \tag{E.5}
\end{equation*}
$$

where tr denotes the trace (sum of diagonal entries) of the square matrix.
Proof. The $n$-th diagonal entry of the $N \times N$ matrix $C^{\prime} C$ is $\sum_{s=1}^{S} c_{s n}^{2}$. Therefore the denominator of (E.4) is

$$
\sum_{s=1}^{S} \sum_{n=1}^{N}\left(f_{s+d}\left(t_{n}\right)-f_{s}\left(t_{n}\right)\right)^{2}=\sum_{n=1}^{N} \sum_{s=1}^{S} c_{s n}^{2}=\operatorname{tr}\left(C^{\prime} C\right),
$$

which is the denominator of (E.5).
Stacking (E.3) into an $N \times 1$ vector and using $G=\left(g_{i}\left(t_{n}\right)\right)$, we obtain

$$
c_{s}=G^{\prime} \gamma_{s}+\epsilon_{s},
$$

where $c_{s}=\left(c_{s n}\right)_{n=1}^{N}, \gamma_{s}=\left(\gamma_{i s}\right)_{i=1}^{I}$, and $\epsilon_{s}=\left(\epsilon_{s}\left(t_{n}\right)\right)_{n=1}^{N}$. Therefore the OLS estimator is $\widehat{\gamma}_{s}=\left(G G^{\prime}\right)^{-1} G^{\prime} c_{s}$ and the $N \times 1$ vector of fitted values is

$$
\widehat{c}_{s}:=G^{\prime} \widehat{\gamma} s=G^{\prime}\left(G G^{\prime}\right)^{-1} G c_{s} .
$$

Stacking this vector for $s=1, \ldots, S$ and taking the transpose, we can define the $S \times N$ matrix of fitted values $\widehat{C}=\left(\widehat{c}_{s n}\right)$ by

$$
\widehat{C}:=C G^{\prime}\left(G G^{\prime}\right)^{-1} G .
$$

By the same argument as the case with the denominator and using the property $\operatorname{tr}(A B)=\operatorname{tr}(B A)$, the numerator of (E.4) becomes

$$
\begin{aligned}
\sum_{s=1}^{S} \sum_{n=1}^{N} \widehat{c}_{s n}^{2} & =\operatorname{tr}\left(\widehat{C}^{\prime} \widehat{C}\right)=\operatorname{tr}\left(\widehat{C} \widehat{C}^{\prime}\right) \\
& =\operatorname{tr}\left(C G^{\prime}\left(G G^{\prime}\right)^{-1} G G^{\prime}\left(G G^{\prime}\right)^{-1} G C^{\prime}\right) \\
& =\operatorname{tr}\left(C G^{\prime}\left(G G^{\prime}\right)^{-1} G C^{\prime}\right) \\
& =\operatorname{tr}\left(G^{\prime}\left(G G^{\prime}\right)^{-1} G C^{\prime} C\right),
\end{aligned}
$$

which is the numerator of (E.5).

## E. 3 Key rate duration matching

This appendix explains the key rate duration matching method of Ho (1992). The key rate duration of a bond with yield curve $y$ and yield change $\Delta$ at time to maturity $t$ is defined by

$$
\operatorname{KRD}(y, t, \Delta):=\frac{P\left(y_{-}\right)-P\left(y_{+}\right)}{2 \Delta P(y)}
$$

where $y_{ \pm}$denotes the yield curve after changing $y(t)$ to $y(t) \pm \Delta$ at a specific term $t$ and linearly interpolating between the adjacent terms. Following the literature, we set the shift to $\Delta=0.01$ ( 100 basis points).

Figure 6 illustrates the procedure for a set of key rates on December 2, 2016. Key rate duration matching amounts to matching the key rate of liabilities at maturities $\left\{t_{j}\right\}_{j=1}^{J}$ using a portfolio of zero-coupon bonds with the same maturities. ${ }^{14}$

[^12]

Figure 6: Key rate perturbations.
Note: The figures show positive and negative perturbations to the yield curve due to a $1 \%$ change in the respective key rate. We linearly interpolate the yields after a change in the key rate to ensure that the yield curve remains continuous. The true yield curve (in blue) is calculated on December 2, 2016.

## E. 4 Sign test

In this section, we test whether the absolute return error of $\mathrm{RI}(1)$ is significantly better compared to HD or KRD. For RI(1) and KRD, we use 5 bonds and for HD we use 3 bonds since the performance with more bonds is comparatively worse. Subsequently, we calculate the 30-day absolute return error (4.2) for non-overlapping sample periods, starting at November 25, 1985. This procedure renders a total of 304 return error observations. Let us denote the return errors for each method by $e_{\mathrm{RI}(1)}, e_{\mathrm{HD}}$ and $e_{\mathrm{KRD}}$. Under the (one-sided) null and alternative hypothesis, we have

$$
\begin{array}{lll}
H_{0}: \operatorname{Pr}\left(e_{\mathrm{RI}(1)}>e_{\mathrm{HD}}\right) \geq 0.5 & \text { vs. } & H_{1}: \operatorname{Pr}\left(e_{\mathrm{RI}(1)}>e_{\mathrm{HD}}\right)<0.5 \\
H_{0}: \operatorname{Pr}\left(e_{\mathrm{RI}(1)}>e_{\mathrm{KRD}}\right) \geq 0.5 & \text { vs. } & H_{1}: \operatorname{Pr}\left(e_{\mathrm{RI}(1)}>e_{\mathrm{KRD}}\right)<0.5 . \tag{E.6b}
\end{array}
$$

The test statistic for the sign test counts the number of positive differences between $e_{\mathrm{RI}(1)}$ and the error term of the alternative method. Under $H_{0}$, this test statistic follows a binomial distribution with success probability $p=0.5$. Using the normal approximation to the binomial distribution, we find $Z$-scores of -5.22 and -9.12 corresponding to the hypotheses (E.6a) and (E.6b). Both test scores are sufficient to reject $H_{0}$ under any conventional significance level.


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[^1]:    ${ }^{1}$ https://www.bankofengland.co.uk/speech/2022/november/ sarah-breeden-speech-at-isda-aimi-boe-on-nbfi-and-leverage
    ${ }^{2}$ https://www.ft.com/content/f9a3adce-1559-4f66-b172-cd45a9fa09d6
    ${ }^{3}$ https://www.economist.com/finance-and-economics/2023/05/03/ what-the-first-republic-deal-means-for-americas-banks

[^2]:    ${ }^{4}$ This statement is similar to the fact that the Black and Scholes (1973) option pricing model has been hugely successful precisely because the model requires only a few assumptions, namely the absence of arbitrage and the stock price following a geometric Brownian motion, and no assumptions on investor preferences are required.

[^3]:    ${ }^{5}$ As we use several different norms in this paper, we use subscripts to distinguish them. An example is the $\ell^{p}$ norm on $\mathbb{R}^{J}$ for $p=1,2$, which we denote by $\|\cdot\|_{p}$.

[^4]:    ${ }^{6}$ Note that we can interchange the order of integration and differentiation using the dominated convergence theorem.

[^5]:    ${ }^{7}$ On the other hand, if $\Delta_{1} \sim \Delta_{2}$, then imposing the constraint $\mathcal{Z}_{1}$ worsens the performance because $V_{I}\left(\mathcal{Z}_{1}\right) \geq V_{I}(\mathcal{Z})$ by Proposition 3.4.

[^6]:    ${ }^{8}$ https://www.federalreserve.gov/data/nominal-yield-curve.htm

[^7]:    ${ }^{9}$ The $\mathrm{RI}(2)$ method is defined only when $J \geq 3$ because otherwise the portfolio constraint $\mathcal{Z}_{2}$ is generally empty.
    ${ }^{10}$ We also considered the relative pricing error $\frac{1}{P\left(x_{s+d}\right)}\left|P\left(x_{s+d}\right)-\sum_{j=1}^{J} z_{s j} P_{j}\left(x_{s+d}\right)\right|$ but it makes no material difference because $P\left(x_{s}\right)$ and $P\left(x_{s+d}\right)$ have the same order of magnitude.

[^8]:    ${ }^{11}$ We chose to estimate the model ourselves instead of using the parameters reported in Ang

[^9]:    et al. (2008, Table III) to better reflect the evolution in yields over the last decade.

[^10]:    12https://home.treasury.gov/resource-center/data-chart-center/interest-rates/ TextView?type=daily_treasury_yield_curve\&field_tdr_date_value=all\&data=yieldAll

[^11]:    ${ }^{13}$ Throughout we ignore the approximation error coming from misspecification of the forward rate model.

[^12]:    ${ }^{14}$ The key rate duration of a zero-coupon bond with maturity $t$ is equal to $t$ and zero otherwise. Since we use linear interpolation after a key rate perturbation to keep the yield curve continuous, the key rate for a zero-coupon bond with maturity $t$ is not exactly equal to $t$ in our application.

